

F.Y.B.COM.
MATHEMATICAL AND
STATISTICAL TECHNIQUE
SEMESTER - II
SUBJECT CODE : UBCOMFSII.7

Prof. Suhas Pednekar

Vice-Chancellor,
University of Mumbai,

Prof. Ravindra D. Kulkarni

Pro Vice-Chancellor,
University of Mumbai,

Prof. Prakash Mahanwar

Director,
IDOL, University of Mumbai,

Programme Co-ordinator & Editor : **Shri Mandar Bhanushe**
Asst. Professor cum Asst. Director in Mathematics
IDOL, University of Mumbai.

Course Co-ordinator : **Smt. Manjiree Gondhalekar**
N.M. College,
Mumbai - 400056

Course Writers : **Shri Mandar Bhanushe**
Asst. Professor cum Asst. Director in Mathematics
IDOL, University of Mumbai.

: **Dr. Abhijit Padte**
L.S. Raheja College,
Mumbai - 400054

: **Dr. Sangeeta Bhat**
S.K. Somaiya College,
Mumbai - 400077

: **Shri Dilip Patil**
J. M. Patel College,
Mumbai - 400062

: **Shri Krishna Nilkumbh**
Chetana College,
Mumbai - 400051

: **Shri Arunachalam S.**
Rizvi College,
Mumbai - 400050

March 2021, Print - I

Published by : Director
Institute of Distance and Open Learning,
University of Mumbai,
Vidyanagari, Mumbai - 400 098.

DTP Composed : Ashwini Arts
Vile Parle (E), Mumbai - 400 099.

Printed by :

CONTENTS

Unit No.	Title	Page No.
SEMESTER - II		
1.	Functions, Derivatives and their Applications	01
2.	Simple Interest and Compound Interest	34
3.	Annuities and EMI	47
4.	Correlation and Regression	62
5.	Time Series	84
6.	Index Numbers	110
7.	Probability Distributions	129



F.Y.B.COM
SEMESTER II
Mathematical and Statistical Techniques-II

[A] MATHEMATICS: (40 marks)

Unit I

Functions Derivatives and Their Applications

Concept of real functions : constant function, linear function, x^2 , e^x , a^x , $\log x$, Demand, Supply, Total Revenue, Average Revenue, Total Cost, Average Cost and Profit function. Equilibrium Point.

Derivative as rate measure.

Derivatives of functions : Constant function, x^n , e^x , a^x , $\log x$

Rules of derivatives : Scalar multiplication, sum, difference, product, quotient, simple problems.

Second Order derivatives.

Applications: Marginal Cost, Marginal Revenue, Elasticity of Demand. Maxima and Minima for functions in Economics and Commerce.

Unit II

Interest and Annuity

Simple Interest and Compound Interest

Interest Compounded more than once a year. Calculations involving upto 4 time periods.

Equated Monthly Instalments (EMI) using reducing & flat interest system. Present value, Future value.

Annuity, Immediate and due : Simple problems with

$$A = P \left(1 + \frac{r}{100} \right)^n \text{ with } n \leq 4$$

[B] STATISTICS: (60 Marks)

Unit III

Bivariate Linear Correlation : Scatter Diagram, Computation of Karl Pearson's Coefficient of Correlation (Case of Bivariate Frequency Table to be excluded), Computation of Spearman's Rank Correlation Coefficient (case of repeated ranks upto 2 repetition only)

Bivariate Linear Regression : Finding Regression lines by method of least squares.

Properties of Regression Coefficients – i) $r = \pm \sqrt{b_{xy} b_{yx}}$ ii) (x, y) is a point of intersection of two regression lines.

Unit IV

Times Series : Concept and Components of time series. Estimation of Trend using Moving Average Method & Least Squares Method (only Linear Trend)

Estimation of Seasonal Component using Simple Arithmetic Mean. (For Trend free data only)

II

Concept of Forecasting using Least Squares Method.

Index Numbers : Concept and uses. Simple and Composite Index Nos. (unweighted, weighted) Laspeyre's Price Index No., Paasche's Price Index No. Fisher's Price Index No., Cost of Living Index No., Real Income, Simple Examples of Wholesale price Index no. (Examples on missing values should not be done)

Unit V

Elementary Probability Theory:

Concept of Random experiment/trial and possible outcomes; Sample Space and Discrete Sample Space; Events and their types, Algebra of Events, Mutually Exclusive and Exhaustive Events, concept of ${}^n C_r$.

Classical definition of Probability, Addition theorem (without proof);

Independence of Events : $P(A \cap B) = P(A) P(B)$

Simple examples.

Random Variable: Probability distribution of a discrete random variable; Expectation and Variance; Simple examples.

Concept of Normal distribution and Standard Normal Variate (SNV), simple examples.



Unit-1

FUNCTIONS, DERIVATIVES AND THEIR APPLICATIONS

Unit Structure :

- 1.0 Objectives
- 1.1 Introduction
- 1.2 Derivatives
- 1.3 Second Order Derivatives
- 1.4 Applications of Derivatives
- 1.5 Maxima and Minima

1.0 OBJECTIVES

After reading this chapter you will be able to recognize.

- 1) Definition of function.
- 2) Standard Mathematical function.
- 3) Definition of derivative.
- 4) Derivatives of standard functions.
- 5) Second order derivatives.
- 6) Application of derivatives.
- 7) Maxima and Minima.

1.1 FUNCTIONS

If $y=f(x)$ is a function then the set of all values of x for which this function is defined is called the domain of the function f . Here x is called an independent variable and y is called the dependent variable. The set of all corresponding values of y for x in the domain is called the range of the function f .

The function f is defined from the domain to the range.

We shall discuss only those functions where the domain and the range are subsets of real numbers. Such functions are called 'real valued functions'.

1.1.1 Standard Mathematical Functions:

(1) Constant function:

The constant function is defined by
 $y=f(x)=C$ where C is a constant.

The constants are denoted by real numbers or alphabets. The graph of a constant function is a straight line parallel to x-axis.

Examples:

$$y=f(x)=5$$

$$y=f(x)=-10$$

$$y=f(x)=K$$

$$y=f(x)=a$$

(2) Linear function:

The linear function is defined by $y=f(x)=ax+b$ where a and b are constants.

Examples: $y=f(x)=2x+5$

$$y=f(x)=-3x+10$$

$$y=f(x)=5x-7$$

(3) Functions with power of x:

A function $f(x)=x^n$ is called power function or function with power of x. Here x is called base and n is called power.

Examples :

$$f(x)=x^2$$

$$f(x)=x^{-5}$$

$$f(x)=x^{-4/3} \quad f(x)=x^{3/2}$$

(4) Exponential functions :

These functions are of the type .

$$f(x)=e^x \text{ and}$$

$$f(x)=a^x, a > 0$$

(5) Logarithmic function : The logarithmic function is defined by

$$y=f(x)=\log_e^x, x > 0$$

1.1.2. Standard functions from Economics :

(1) Demand : It refers to the quantity of a product is desired by the buyers . The demand depends on the price. Therefore, there is a relationship between the price and the quantity demanded. Such relationship is called a demand function.

Hence the demand function is defined as

$$D=g(p) \text{ where } D=\text{demand and } p=\text{price} .$$

Here demand is a dependent variable and the price is an independent variable.

For example, $D = 50 + 4p - 3p^2$

(2) Supply : It refers to the quantity of a product , the market can offer . The supply depends on the price. Therefore, There is a relationship between the price and the quantity supplied. Such relationship is called a supply function.

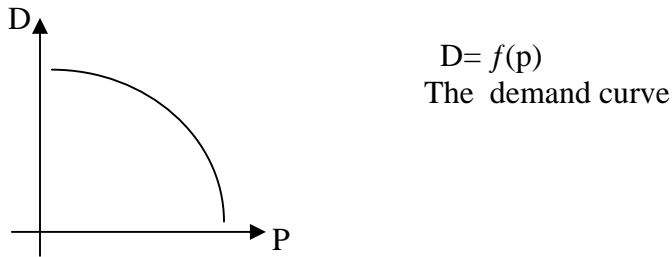
Hence the supply function is defined as $S = f(p)$ where $S =$ supply and $p =$ price.

Here supply is a dependent variable and price is an independent variable.

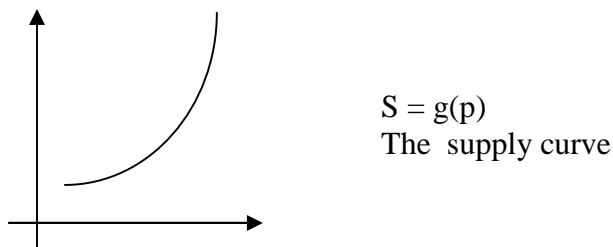
For example, $S = 2p^2 - 6p + 25$

(3) Break-even Point : Equilibrium point.

(i) By the law of demand, the demand decreases when the price increases, the demand curve is a decreasing curve as shown in the figure:

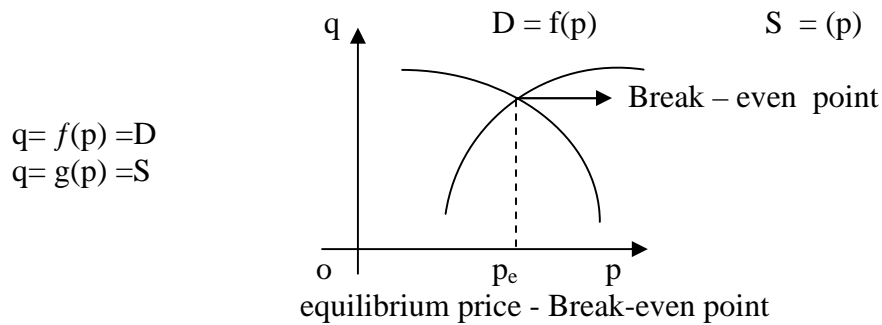


(ii) By the law of supply, the supply increases when the price increases, the supply curve is an increasing curve as shown in the figure.



(iii) The demand and supply curves $D = f(p)$ and $S = g(p)$ are intersecting at a point . The point of intersection of the demand and supply curves represents that specific price at which the demand and supply are equal. This point is called the Break-even point or equilibrium point. The corresponding price at which this point occurs is called an equilibrium price and is denoted by p_e

At equilibrium price, the amount of goods supplied is equal to the amount of goods demanded.

**(4) The total cost function :**

The total cost function or cost function is denoted by C and it is expressed in terms of x . If C is the cost of producing x units of a product, then C is generally a function of x and is called the total cost function.

$$\text{i.e. } C = f(x)$$

$$\text{For example, } C = 2x^2 - 5x + 10$$

(5) Average cost function :

The ratio between the cost function and the number of units produced is called average cost function. i.e. $AC = \frac{C}{x}$

$$\text{For example, } AC = \frac{x^2 + 2x + 5}{x}$$

(6) Total Revenue function :

The total revenue function is defined as in terms of the demand and the price per item. If D units are demanded with the selling price of p per unit, then the total revenue function R is given by

$$R = p \times D \text{ where } p = \text{price and } D = \text{demand}$$

For example,

$$\text{If } D = p^2 + 2p + 3 \text{ then } R = p \times (p^2 + 2p + 3) = p^3 + 2p^2 + 3p$$

(7) Average revenue :

Average revenue is defined as the ratio between the revenue and the demand and is denoted by AR .

$$\text{i.e., } AR = R/D \quad AR = p \times D/D \quad (\text{as } R = p \times D)$$

$$\therefore AR = p$$

\therefore Average revenue is nothing but the selling price per unit.

(8) The Profit function :

The profit function or the total profit function is denoted by P and is defined by the difference between the total revenue and the total cost.

$$\therefore \text{ Total Profit} = \text{Total Revenue} - \text{Total cost}$$

$$\text{i.e. } P = R - C$$

Example 1:

Find the total profit function if the cost function $C = 40 + 15x - x^2$,
 x = number of items produced and the demand function is $p = 200 - x^2$

Solution :

$$\begin{aligned}
 \text{Given } C &= 40 + 15x - x^2 \\
 p &= 200 - x^2 \\
 R &= p \times D \quad (D=x) \\
 &= (200 - x^2) x \\
 \therefore R &= 200x - x^3 \\
 \text{Profit} &= \text{Revenue} - \text{Cost} \\
 P &= R - C \\
 &= (200x - x^3) - (40 + 15x - x^2) \\
 &= 200x - x^3 - 40 - 15x + x^2 \\
 P &= 185x + x^2 - x^3 - 40
 \end{aligned}$$

Example 2: The total cost function is $C = 20 - 3x^2$ and the demand function is $p = 5 + 6x$. Find the profit when $x = 100$.

Solution :

$$\begin{aligned}
 \text{Given : } C &= 20 - 3x^2 \\
 R &= p \times D \quad (D=x) \\
 &= p \times x \\
 &= (5 + 6x) x \\
 &= 5x + 6x^2 \\
 \text{Profit} &= \text{Revenue} - \text{Cost} \\
 &= R - C \\
 &= (5x + 6x^2) - (20 - 3x^2) \\
 &= 5x + 6x^2 - 20 + 3x^2 \\
 &= 5x + 9x^2 - 20 \\
 \text{When } x &= 100, P = 5(100) + 9(100)^2 - 20 \\
 &= 500 + 90000 - 20 \\
 &= 90480.
 \end{aligned}$$

1.2 DERIVATIVES

1.2.1 Derivative as rate measure :

Definition : Let $y = f(x)$ be the given function .

$$\text{If } \lim_{h \rightarrow 0} \frac{[f(x+h) - f(x)]}{h} \text{ exists ,}$$

then we say that the function $f(x)$ has derivative at x and is denoted by $f'(x)$.

$$\text{i.e. , } f'(x) = \lim_{h \rightarrow 0} \frac{[f(x+h) - f(x)]}{h}$$

The rate of change is called the " derivative " of $y = f(x)$ with respect to x and is denoted by $\frac{dy}{dx}$ or $f'(x)$.

$\frac{dy}{dx}$ = the rate of change of y with respect to x or the derivative of y with respect to x .

Note : (1) Derivative means " rate of change "

(2) The process of finding the derivative of a function is called " differentiation".

$\frac{dC}{dx}$ = the rate of change cost with respect to x .

For example $\frac{dD}{dp}$ = the rate of change of demand with respect to p .

1.2.2 Derivatives of Standard functions :

(1) If $y = x^n$, where n is a real number, then

$$\frac{dy}{dx} = n x^{n-1}$$

i.e., $\frac{dy}{dx} = \frac{d(x^n)}{dx} = n x^{n-1}$

(2) If $y = C$, where C is a constant,

then $\frac{dy}{dx} = 0$

i.e., $\frac{dy}{dx} = \frac{d(C)}{dx} = 0$

(3) If $y = e^x$, then $\frac{dy}{dx} = e^x$

i.e., $\frac{dy}{dx} = \frac{d(e^x)}{dx} = e^x$

(4) If $y = a^x$, where a is a positive real number, then

$$\frac{dy}{dx} = a^x \log a$$

i.e., $\frac{dy}{dx} = \frac{d(a^x)}{dx} = a^x \log a$

(5) If $y = \log x$, then $\frac{dy}{dx} = \frac{1}{x}$

where $x > 0$

i.e., $\frac{dy}{dx} = \frac{d(\log x)}{dx} = \frac{1}{x}$

Examples :

(1) $y = x = x^1 \therefore \frac{dy}{dx} = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1$

(2) $y = x^4 \therefore \frac{dy}{dx} = 4 \cdot x^{4-1} = 4 \cdot x^3$

(3) $y = x^{10} \therefore \frac{dy}{dx} = 10 \cdot x^{10-1} = 10x^9$

(4) $y = \frac{1}{x} = x^{-1} \therefore \frac{dy}{dx} = -1 \cdot x^{-1-1} = -1x^{-2} = -\frac{1}{x^2}$

$$(5) y = \frac{1}{x^3} = x^{-3} \therefore \frac{dy}{dx} = -3x^{-3-1} = -3x^{-4} = \frac{-3}{x^4}$$

$$(6) y = \sqrt{x} = x^{1/2} \therefore \frac{dy}{dx} = \frac{1}{2} x^{1/2-1} = \frac{1}{2} x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$$

$$(7) y = x^{5/2} \therefore \frac{dy}{dx} = \frac{5}{2} x^{5/2-1} = \frac{5}{2} x^{3/2}$$

$$(8) y = x^{-3/2} \\ \therefore \frac{dy}{dx} = -\frac{3}{2} x^{-3/2-1} = -\frac{3}{2} x^{-5/2}$$

$$(9) y = x^{-7/2} \\ \frac{dy}{dx} = -\frac{7}{2} x^{-7/2-1} \\ = -\frac{7}{2} x^{-9/2}$$

$$(10) y = 5, 5 \text{ is a constant} \\ \therefore \frac{dy}{dx} = 0$$

$$(11) y = K, \frac{dy}{dx} = 0$$

$$(12) y = \log 2, \frac{dy}{dx} = 0$$

$$(13) y = -10 \therefore \frac{dy}{dx} = 0$$

$$(14) y = e^x \therefore \frac{dy}{dx} = e^x$$

$$(15) y = 2^x \therefore \frac{dy}{dx} = 2^x \log 2$$

$$(16) y = 10^x \therefore \frac{dy}{dx} = 10^x \log 10$$

$$(17) y = \log x \therefore \frac{dy}{dx} = \frac{1}{x}$$

Exercise: 1.1

Find $\frac{dy}{dx}$ for the following :

$$(1) y = x^6$$

$$(2) y = \frac{1}{x^2}$$

$$(3) y = x^{7/2}$$

- (4) $y = x^{-5/2}$
 (5) $y = 3$
 (6) $y = \log 10$
 (7) $y = -8$
 (8) $y = 4^x$
 (9) $y = 9^x$
 (10) $y = 15^{-x}$

Answers :

- (1) $6x^5$ (2) $\frac{-2}{x^3}$ (3) $\frac{7x^{5/2}}{2}$ (4) $-\frac{5}{2}x^{-7/2}$ (5) 0 (6) 0 (7) 0 (8) $4^x \log 4$
 (9) $9^x \log 9$ (10) $15^x \log 15$

1.2.3 Rules of derivatives :

Rule : 1 Addition Rule (or) Sum rule :

If $y = u + v$ where u and v are differentiable functions of x then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

i.e. $\frac{dy}{dx} = \frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

Examples:

- (1) If $y = x^2 + e^x$, find $\frac{dy}{dx}$

Solution : Given : $y = x^2 + e^x$

$$\frac{dy}{dx} = \frac{d(x^2 + e^x)}{dx}$$

$$= \frac{d(x^2)}{dx} + \frac{d(e^x)}{dx}$$

$$\therefore \frac{dy}{dx} = 2x + e^x$$

- (2) If $y = x^{10} + \log x$, find $\frac{dy}{dx}$

Solution: Given : $y = x^{10} + \log x$

$$\frac{dy}{dx} = \frac{d[x^{10} + \log x]}{dx}$$

$$= \frac{d(x^{10})}{dx} + \frac{d(\log x)}{dx}$$

$$\therefore \frac{dy}{dx} = 10x^9 + \frac{1}{x}$$

Rule:2 Subtraction Rule (or) Difference rule:

If $y = u - v$ where u and v are differentiable functions of x then

$$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$$

$$\text{i.e., } \boxed{\frac{dy}{dx} = \frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}}$$

Examples :

(1) If $y = x^5 - 2^x$, find $\frac{dy}{dx}$

Solution : Given : $y = x^5 - 2^x$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{d}{dx}(x^5 - 2^x) \\ &= \frac{d}{dx}(x^5) - \frac{d}{dx}(2^x) \end{aligned}$$

$$\therefore \frac{dy}{dx} = 5x^4 - 2^x \log 2$$

(2) If $y = 100 - \log x$, find $\frac{dy}{dx}$

Solution : Given : $y = 100 - \log x$

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(100 - \log x) \\ &= \frac{d}{dx}(100) - \frac{d}{dx}(\log x) \\ &= 0 - \frac{1}{x} \\ \therefore \frac{dy}{dx} &= -\frac{1}{x} \end{aligned}$$

Rule : 3 Product Rule :

If $y = uv$ where u and v are differentiable functions of x ,

$$\text{then } \frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$\text{i.e., } \boxed{\frac{dy}{dx} = \frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}}$$

Examples :

(1) If $y = x^4 \log x$, find $\frac{dy}{dx}$

Solution : Given $y = x^4 \log x$
 $u = x^4, v = \log x$

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 \log x)$$

$$\begin{aligned}
&= x^4 \frac{d(\log x)}{dx} + \log x \frac{d(x^4)}{dx} \\
&= x^4 (1/x) + \log x (4x^3) \\
&= x^3 + 4x^3 \log x \\
\therefore \frac{dy}{dx} &= x^3 [1 + 4 \log x]
\end{aligned}$$

(2) If $y = x^2 e^x$, find $\frac{dy}{dx}$

Solution : Given $y = x^2 e^x$

$$\begin{aligned}
\frac{dy}{dx} &= x^2 \frac{d(e^x)}{dx} + e^x \frac{d(x^2)}{dx} \\
&= x^2 e^x + e^x (2x) \\
\therefore \frac{dy}{dx} &= e^x [x^2 + 2x]
\end{aligned}$$

Rule :4 Quotient Rule :

If $y = \frac{u}{v}$, $v \neq 0$ where u and v are differentiable functions of x , then

$$\frac{dy}{dx} = \left[\frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \right]$$

Examples :

(1) If $y = \frac{x+4}{\log x}$, find $\frac{dy}{dx}$

Solution: Given $y = \frac{x+4}{\log x}$

$$\begin{aligned}
\text{Here } u &= x+4 \\
v &= \log x
\end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{\log x \frac{d(x+4)}{dx} - (x+4) \frac{d(\log x)}{dx}}{(\log x)^2}$$

$$= \frac{(\log x) (1+0) - (x+4) (1/x)}{(\log x)^2}$$

$$\therefore \frac{dy}{dx} = \frac{(\log x) - (x+4) (1/x)}{(\log x)^2}$$

(2) If $y = \frac{e^x+5}{x^6-10}$, find $\frac{dy}{dx}$

Solution :

$$\text{Given : } y = \frac{e^x+5}{x^6-10}$$

Here $u = e^x + 5$; $v = x^6 - 10$

$$\therefore \frac{dy}{dx} = \frac{(x^6 - 10) \frac{d}{dx}(e^x + 5) - (e^x + 5) \frac{d}{dx}(x^6 - 10)}{(x^6 - 10)^2}$$

$$\therefore \frac{dy}{dx} = \frac{(x^6 - 10)(e^x) - (e^x + 5)(6x^5)}{(x^6 - 10)^2}$$

(3) If $y = \frac{x^3 - 1}{x^3 + 1}$, find $\frac{dy}{dx}$

Solution :

$$\text{Given : } y = \frac{x^3 - 1}{x^3 + 1}$$

$$\text{Here } u = x^3 - 1$$

$$v = x^3 + 1$$

$$\therefore \frac{dy}{dx} = \frac{(x^3 + 1) \frac{d}{dx}(x^3 - 1) - (x^3 - 1) \frac{d}{dx}(x^3 + 1)}{(x^3 + 1)^2}$$

$$= \frac{(x^3 + 1)(3x^2) - (x^3 - 1)(3x^2)}{(x^3 + 1)^2}$$

$$= \frac{3x^2[(x^3 + 1) - (x^3 - 1)]}{(x^3 + 1)^2}$$

$$= \frac{3x^2[x^3 + 1 - x^3 + 1]}{(x^3 + 1)^2}$$

$$= \frac{3x^2 [2]}{(x^3 + 1)^2}$$

$$\frac{dy}{dx} = \frac{6x^2}{(x^3 + 1)^2}$$

Rule 5 : Scalar multiplication rule or constant multiplied by a function rule.

If $y = cu$, c is a constant , where u is a differentiable function of x , then

$$\frac{dy}{dx} = c \frac{du}{dx}$$

$$\text{i.e., } \boxed{\frac{dy}{dx} = \frac{d}{dx}(cu) = c \frac{du}{dx}}$$

Examples :

(1) If $y = 5x^3$, find $\frac{dy}{dx}$

Solution : Given : $y = 5x^3$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(5x^3)$$

$$\frac{dx}{dx} = 5 \frac{d}{dx} (x^3) = 5(3x^2) = 15x^2$$

(2) If $y = 10 \log x$, find $\frac{dy}{dx}$

Solution : Given : $y = 10 \log x$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{d}{dx} (10 \log x) \\ &= 10 \frac{d}{dx} (\log x) \\ &= 10 (1/x) \\ &= \frac{10}{x} \end{aligned}$$

1.2.4 List of formulae :

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
1. x^n	nx^{n-1}
2. $C, C = \text{constant}$	0
3. e^x	e^x
4. a^x	$a^x \log a$
5. $\log x$	$1/x$
6. x	1
7. \sqrt{x}	$1/2\sqrt{x}$
8. $1/x$	$-1/x^2$

1.2.5 List of Rules :

- (1) $\frac{d}{dx} (u+v) = \frac{du}{dx} + \frac{dv}{dx}$
- (2) $\frac{d}{dx} (u-v) = \frac{du}{dx} - \frac{dv}{dx}$
- (3) $\frac{d}{dx} (uv) = u \frac{dv}{dx} + v \frac{du}{dx}$
- (4) $\frac{d}{dx} (u/v) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$
- (5) $\frac{d}{dx} (cu) = c \frac{du}{dx}$, $c = \text{constant}$.

1.2.6 Examples :

Find $\frac{dy}{dx}$ for each of the following :

Ex: (1) $y = x^6 + 4e^x + \log x + 10$

Solution:

$$\frac{dy}{dx} = \frac{d}{dx} (x^6 + 4e^x + \log x + 10)$$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx}(x^6) + \frac{d}{dx}(4e^x) + \frac{d}{dx}(\log x) + \frac{d}{dx}(10) \\
 &= \frac{d}{dx}(x^6) + 4\frac{d}{dx}(e^x) + \frac{d}{dx}(\log x) + \frac{d}{dx}(10) \\
 &= 6x^5 + 4e^x + 1/x + 0 \\
 \therefore \frac{dy}{dx} &= 6x^5 + 4e^x + 1/x
 \end{aligned}$$

Ex : (2) $y = 5x^4 - 3e^x + 4\sqrt{x} + 2^x$

Solution :

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx}(5x^4 - 3e^x + 4\sqrt{x} + 2^x) \\
 &= \frac{d}{dx}(5x^4) - \frac{d}{dx}(3e^x) + \frac{d}{dx}(4\sqrt{x}) + \frac{d}{dx}(2^x) \\
 &= 5\frac{d}{dx}(x^4) - 3\frac{d}{dx}(e^x) + 4\frac{d}{dx}(\sqrt{x}) + \frac{d}{dx}(2^x) \\
 &= 5(4x^3) - 3e^x + 4(1/2\sqrt{x}) + 2^x \log_2 \\
 &= 20x^3 - 3e^x + 2/\sqrt{x} + 2^x \log_2
 \end{aligned}$$

Ex: (3) $y = x^{3/2} + 4 \log x - 10x^2 + 15$

Solution:

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{d}{dx}(x^{3/2} + 4 \log x - 10x^2 + 15) \\
 &= \frac{d}{dx}(x^{3/2}) + \frac{d}{dx}(4 \log x) - \frac{d}{dx}(10x^2) + \frac{d}{dx}(15) \\
 &= \frac{3}{2}x^{3/2-1} + 4\left(\frac{1}{x}\right) - 10(2x) + 0 \\
 \frac{dy}{dx} &= \frac{3}{2}x^{1/2} + \frac{4}{x} - 20x.
 \end{aligned}$$

Ex: (4) $y = (x + e^x)(5 + \log x)$

Solution: Here $u = x + e^x$
 $v = 5 + \log x$

$$\begin{aligned}
 \frac{dy}{dx} &= (x + e^x) \frac{d}{dx}(5 + \log x) + (5 + \log x) \frac{d}{dx}(x + e^x) \\
 &= (x + e^x)(0 + 1/x) + (5 + \log x)(1 + e^x) \\
 \therefore \frac{dy}{dx} &= (x + e^x)(1/x) + (5 + \log x)(1 + e^x)
 \end{aligned}$$

Ex: (5) $y = (x^{10}) (10^x)$

Solution : Here $u = x^{10}$, $v = 10^x$

$$\therefore \frac{dy}{dx} = x^{10} \frac{d}{dx} (10^x) + 10^x \frac{d}{dx} (x^{10}) = x^{10} (10^x \log 10) + 10^x (10x^9)$$

Ex: (6) $y = (\sqrt{x} + e^x) (2x^3 + 7)$

Solution: Here $u = \sqrt{x} + e^x$, $v = 2x^3 + 7$

$$\frac{dy}{dx} = (\sqrt{x} + e^x) \frac{d}{dx} (2x^3 + 7) + (2x^3 + 7) \frac{d}{dx} (\sqrt{x} + e^x)$$

$$\therefore \frac{dy}{dx} = (\sqrt{x} + e^x) (6x^2) + (2x^3 + 7) (1/2\sqrt{x} + e^x)$$

Ex: (7) $y = \frac{x^2 + 5x + 6}{x + 7}$

Solution : Here $u = x^2 + 5x + 6$
 $v = x + 7$

$$\therefore \frac{dy}{dx} = \frac{(x + 7) \frac{d}{dx} (x^2 + 5x + 6) - (x^2 + 5x + 6) \frac{d}{dx} (x + 7)}{(x + 7)^2}$$

$$= \frac{(x + 7) (2x + 5) - (x^2 + 5x + 6) (1)}{(x + 7)^2}$$

$$= \frac{2x^2 + 19x + 35 - x^2 - 5x - 6}{(x + 7)^2}$$

$$\therefore \frac{dy}{dx} = \frac{x^2 + 14x + 29}{(x + 7)^2}$$

Ex : 8 $y = \frac{10 e^x + 5 \log x}{x^3 + 12}$

Solution : Here $u = 10e^x + 5 \log x$
 $v = x^3 + 12$

$$\therefore \frac{dy}{dx} = \frac{(x^3 + 12) \frac{d}{dx} (10e^x + 5 \log x) - (10e^x + 5 \log x) \frac{d}{dx} (x^3 + 12)}{(x^3 + 12)^2}$$

$$\frac{dy}{dx} = \frac{(x^3 + 12) (10e^x + 5/x) - (10e^x + 5 \log x) (3x^2)}{(x^3 + 12)^2}$$

Ex:9 $y = \frac{4^x + 6}{2x^2 + 5}$

Solution : Here $u = 4^x + 6$, $v = 2x^2 + 5$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{(2x^2+5) \frac{d}{dx}(4^x+6) - (4^x+6) \frac{d}{dx}(2x^2+5)}{(2x^2+5)^2} \\ &= \frac{(2x^2+5)(4^x \log 4) - (4^x+6)(4x)}{(2x^2+5)^2} \end{aligned}$$

Ex: 10 $y = 2\sqrt{x} + 16e^x + (6^x) + 20x$

Solution:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(2\sqrt{x} + 16e^x + 6^x + 20x) \\ &= 2 \left(\frac{1}{2\sqrt{x}} \right) + 16e^x + 6^x \log 6 + 20 \end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{x}} + 16e^x + 6^x \log 6 + 20$$

Exercise : 1.2

Differentiate the following with respect to x .

(1) $y = x^8 - 6e^x + 4x^{3/2} - 3x^2 + 5$

(2) $y = 6 \log x - 3^x + 2e^x + 10\sqrt{x} + 2$

(3) $y = 5x^4 - 12x^3 + 18e^x + 10^x - 25$

(4) $y = 8^x(5x^3 + 3x + 1)$

(5) $y = (10x^2 + 2x + 5)(\sqrt{x} + e^x)$

(6) $y = (2x^3 + 3x^2)(5 \log x + 14)$

(7) $y = (x + \log x)(x^5 - 4x^2 + 10)$

(8) $y = (8x^5 - 6x^{5/2} + 1)(40\sqrt{x} + 2e^x)$

(9) $y = (e^x + 2 \log x + 2)(6^x + 2x^2 + 5)$

(10) $y = \frac{x^2 + 1}{x^4 - 1}$

(11) $y = \frac{2^x + 4\sqrt{x}}{2e^x + 5}$

(12) $y = \frac{x^3 - x^2 + 2}{x^2 - 4}$

(13) $y = \frac{x + \sqrt{x}}{\sqrt{x} - 1}$

(14) $y = \frac{3 \log x + 5}{x^5 + 2x}$

(15) $y = \frac{e^x - \sqrt{x}}{2\sqrt{x} + 1}$

Answers :

- (1) $8x^7 - 6e^x + 6x^{1/2} - 6x$
- (2) $6/x - 3^x \log 3 + 2e^x + 5/\sqrt{x}$
- (3) $20x^3 - 36x^2 + 18e^x + 10^x \log 10$
- (4) $8^x (15x^2 + 3) + (5x^3 + 3x + 1) (8^x \log 8)$
- (5) $(10x^2 + 2x + 5) (1/2\sqrt{x} + e^x) + (\sqrt{x} + e^x) (20x + 2)$
- (6) $(2x^3 + 3x^2) (5/x) + (5 \log x + 14) (6x^2 + 6x)$
- (7) $(x + \log x) (5x^4 - 8x) + (x^5 - 4x^2 + 10) (1 + 1/x)$
- (8) $(8x^5 - 6x^{5/2} + 1) (20/\sqrt{x} + 2e^x) + (40\sqrt{x} + 2e^x) (40x^4 - 15x^{3/2})$
- (9) $(e^x + 2 \log x + 2) (6^x \log 6 + 4x) + (6^x + 2x^2 + 5) (e^x + 2/x)$
- (10) $\frac{-2x^5 - 4x^3 - 2x}{(x^4 - 1)^2}$
- (11) $\frac{(2e^x + 5) (2^x \log 2 + 2/\sqrt{x}) - (2^x + 4\sqrt{x}) (2e^x)}{(2e^x + 5)^2}$
- (12) $\frac{x^4 - 12x^2 + 4x}{(x^2 - 4)^2}$
- (13) $\frac{\sqrt{x} - 1 - (1+x)/2\sqrt{x}}{(\sqrt{x} - 1)^2}$
- (14) $\frac{(x^5 + 2x) (3/x) - (3 \log x + 5) (5x^4 + 2)}{(x^5 + 2x)^2}$
- (15) $\frac{(2\sqrt{x} + 1) (e^x - 1/2\sqrt{x}) - (e^x - \sqrt{x}) (1/\sqrt{x})}{(2\sqrt{x} + 1)^2}$

1.3 SECOND ORDER DERIVATIVES

If $y = f(x)$ is differentiable function with respect to x , then $\frac{dy}{dx} = f'(x)$ is called first order derivative of y with respect to x and dx

$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$ is called the second order derivative of y with respect to X

Notation :

(i) First order derivative :

$$\frac{dy}{dx} = f'(x) = y_1 = y'$$

(ii) Second order derivative :

$$\frac{d^2y}{dx^2} = f''(x) = y_2 = y''$$

1.3.1 Examples :

(1) If $y = x^3 - 6x^2 + 19x + 100$, find $\frac{d^2y}{dx^2}$

Solution :

$$\text{Given : } y = x^3 - 6x^2 + 19x + 100$$

$$\frac{dy}{dx} = 3x^2 - 12x + 19$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(3x^2 - 12x + 19)$$

$$\frac{d^2y}{dx^2} = 6x - 12$$

(2) If $y = e^x + 2x^3 + 5x^2 + 4$, find $\frac{d^2y}{dx^2}$

Solution :

$$\text{Given : } y = e^x + 2x^3 + 5x^2 + 4$$

$$\frac{dy}{dx} = e^x + 6x^2 + 10x$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(e^x + 6x^2 + 10x)$$

$$= \frac{d}{dx}(e^x + 6x^2 + 10x)$$

$$\frac{d^2y}{dx^2} = e^x + 12x + 10$$

(3) If $y = f(x) = x^5 - 6x^3 + 2x^2 + 10x + 5$, find $f''(x)$

Solution:

$$\text{Given } f(x) = x^5 - 6x^3 + 2x^2 + 10x + 5$$

$$f'(x) = 5x^4 - 18x^2 + 4x + 10$$

$$f''(x) = 20x^3 - 36x + 4$$

(4) If $y = x + 1/x$, $x \neq 0$, find d^2y/dx^2

Solution: Given: $y = x + 1/x$

$$dy/dx = 1 + (-1/x^2) = 1 - 1/x^2$$

$$\frac{d^2y}{dx^2} = 0 - (-2/x^3) = \frac{2}{x^3}$$

Exercise:9.3

Find $\frac{d^2y}{dx^2}$ for each of the following:

(1) $y = 8x^5 - 16x^4 + 4x^3 + x + 2$

(2) $y = 2x^3 - 5x^2 + 12x + 15$

(3) $y = x + 25/x$

(4) $y = 2x^2 + e^x + 5x + 12$

(5) $y = x^2 + x + \log x$

Answers:

1) $160x^3 - 192x^2 + 24x$

2) $12x - 10$

3) $\frac{50}{x^3}$

4) $4 + e^x$

5) $2 - \frac{1}{x^2}$

1.4 APPLICATIONS OF DERIVATIVES

Applications to Economics:

1.4.1 The Total Cost function:

(i) Total cost function $C = f(x)$

(ii) Average cost function $AC = \frac{C}{x}$

(iii) Marginal cost function:

The rate of change of cost with respect to the number of units produced is called the Marginal cost and is denoted by MC.

$$\text{i.e., } MC = \frac{dC}{dx}$$

1.4.2 The Total Revenue Function:

(i) Total Revenue Function $R = p \times D$

(ii) Average Revenue Function $AR = p$

(iii) Marginal Revenue function:

The rate of change of total revenue with respect to the demand D is called

the Marginal revenue function and is denoted by MR.

$$\text{i.e. } MR = \frac{dR}{dD}$$

1.4.3 Elasticity:

Let D be the demand and p be the price. The quantity $-\frac{p}{D} \frac{dD}{dp}$ is

called elasticity of demand with respect to the price and is denoted by η ($\eta = \text{eta}$: Greek alphabet).

$$\text{i.e., } \eta = -\frac{p}{D} \frac{dD}{dp}$$

If $\eta = 0$, demand D is Constant and the demand is said to be perfectly elastic.

If $0 < \eta < 1$, the demand is said to be inelastic

If $\eta = 1$, the demand is directly proportional to the price.

If $\eta > 1$, the demand is said to be elastic.

1.4.4 Relation between the Marginal Revenue and elasticity of demand.

Let $R =$ Total revenue
 $p =$ price
 $D =$ demand

$$\therefore R = pD$$

$$MR = \frac{dR}{dD}$$

$$= \frac{d}{dD}(pD)$$

$$= p \frac{d}{dD}(D) + D \frac{d}{dD}(p) \text{ (by product rule)}$$

$$= p(1) + D \frac{dp}{dD}$$

$$MR = p + D \frac{dp}{dD}$$

$$\eta = -\frac{p}{D} \frac{dD}{dp}$$

$$\therefore D \frac{dp}{dD} = \frac{-p}{\eta}$$

$$MR = p + \left(\frac{-p}{\eta} \right)$$

$$MR = p \left[1 - \frac{1}{\eta} \right]$$

$$MR = AR \left[1 - \frac{1}{\eta} \right] \quad (AR = p)$$

1.4.5 Examples:

Ex: (1) The cost of producing x items is given by

$2x^2 + 5x + 20$. Find the total cost, average cost and marginal cost when $x = 10$.

Solution:

$$\text{Let } C = f(x) = 2x^2 + 5x + 20$$

$$\text{AC} = \frac{C}{x} = \frac{2x^2 + 5x + 20}{x}$$

$$\text{MC} = \frac{dC}{dx}$$

$$= \frac{d}{dx} (2x^2 + 5x + 20)$$

$$\text{MC} = 4x + 5$$

when $x = 10$

$$C = 2(10)^2 + 5(10) + 20$$

$$C = 270$$

$$\text{AC} = \frac{C}{x} = \frac{270}{10} = 27$$

$$\text{MC} = 4(10) + 5 = 45$$

Ex: (2)

The demand function is given by $p = 50 + 6D + 4D^2$. Find the total revenue, average revenue and the marginal revenue when the demand is 5 units.

Solution:

$$\text{Given : } p = 50 + 6D + 4D^2$$

$$R = p \times D$$

$$= (50 + 6D + 4D^2)(D)$$

$$R = 50D + 6D^2 + 4D^3$$

$$\text{AR} = p = 50 + 6D + 4D^2$$

$$\text{MR} = \frac{dR}{dD}$$

$$= \frac{d}{dD} (50D + 6D^2 + 4D^3)$$

$$MR = 50 + 12D + 12D^2$$

When $D = 5$

$$R = 50(5) + 6(5)^2 + 4(5)^3$$

$$= 250 + 150 + 500$$

$$R = 900$$

$$AR = 50 + 6(5) + 4(5)^2$$

$$= 50 + 30 + 100$$

$$= 180$$

$$MR = 50 + 12(5) + 12(5)^2$$

$$= 50 + 60 + 300$$

$$= 410$$

Ex : (3)

The total revenue function is given by $R = 20D - D^2$, $D =$ Demand. Find the demand function. Also find AR when $MR = 0$.

Solution :

$$\text{Given } R = 20D - D^2$$

$$\rho D = 20D - D^2 \quad (\because R = \rho D)$$

$$\therefore \rho = \frac{20D - D^2}{D}$$

$$\rho = 20 - D$$

\therefore The demand function is $\rho = 20 - D$.

$$\text{Now, } MR = \frac{dR}{dD}$$

$$= \frac{d}{dD} (20D - D^2)$$

$$MR = 20 - 2D$$

Given that $MR = 0$.

$$\therefore 0 = 20 - 2D$$

$$2D = 20$$

$$\therefore D = 10$$

$$AR = p = 20 - D$$

$$\therefore AR = 20 - 10$$

$$AR = 10$$

Ex : (4)

The demand function is given by

$$D = 25 - 2p - p^2$$

Find the elasticity of demand when the price is 4.

Solution :

$$\text{Given : } D = 25 - 2p - p^2$$

$$\therefore \frac{dD}{dp} = 0 - 2 - 2p$$

$$\frac{dD}{dp} = -2 - 2p$$

$$\therefore \eta = \frac{-p}{D} \frac{dD}{dp}$$

$$= \frac{(-p)}{25 - 2p - p^2} (-2 - 2p)$$

$$\eta = \frac{p(2 + 2p)}{25 - 2p - p^2}$$

When $p = 4$,

$$\eta = \frac{4(2 + 8)}{25 - 8 - 16}$$

$$= \frac{4(10)}{1}$$

$$\therefore \eta = 40$$

Ex: 5

The demand function is given by $D = \frac{p+3}{2p-1}$

where D = Demand and p = price. Find the elasticity of demand when the price is 8.

Solution:

$$\text{Given } D = \frac{p+3}{2p-1}$$

$$\therefore \frac{dD}{dp} = \frac{(2p-1) \frac{d}{dp}(p+3) - (p+3) \frac{d}{dp}(2p-1)}{(2p-1)^2}$$

$$= \frac{(2p-1)(1) - (p+3)(2)}{(2p-1)^2}$$

$$= \frac{2p-1-2p-6}{(2p-1)^2}$$

$$\frac{dD}{dp} = \frac{-7}{(2p-1)^2}$$

$$\therefore \eta = -\frac{p}{D} \frac{dD}{dp}$$

$$= \frac{(-p)}{[(p+3)/(2p-1)]} \left[\frac{-7}{(2p-1)^2} \right]$$

$$= \frac{7p(2p-1)}{(p+3)(2p-1)^2}$$

$$\eta = \frac{7p}{(p+3)(2p-1)}$$

When $p = 8$

$$\eta = \frac{7(8)}{(8+3)[2(8)-1]}$$

$$= \frac{56}{(11)(15)}$$

$$= \frac{56}{165}$$

$$\eta = 0.34$$

Ex : (6) If MR = 45, AR = 75, Find η

Solution:

$$\text{Given } MR = 45$$

$$AR = 75$$

$$\eta = ?$$

$$\therefore MR = AR [1 - 1/\eta]$$

$$45 = 75 [1 - 1/\eta]$$

$$\frac{45}{75} = 1 - 1/\eta$$

$$\frac{3}{5} = 1 - 1/\eta$$

$$0.6 = 1 - 1/\eta$$

$$1/\eta = 1 - 0.6$$

$$1/\eta = 0.4$$

$$\eta = 1/0.4$$

$$= 2.5$$

Ex : (7)

If AR = 95 and $\eta = 7/2$, Find MR.

$$\text{Solution : Given } AR = 95$$

$$\eta = 7/2 = 3.5$$

$$MR = ?$$

$$MR = AR [1 - 1/\eta]$$

$$= 95 [1 - 1/3.5]$$

$$= 95 [1 - 0.29]$$

$$= 95 [0.71]$$

$$MR = 67.45$$

Exercise : 1.4

- (1) The cost of producing x items is given by $x^3 + 4x + 15$. Find the total cost, average cost and marginal cost when $x = 6$.
- (2) The total cost function is given by $C = x^3 + 2x^2 + 5x + 30$. Find the total cost, average cost and marginal cost when $x = 10$.
- (3) The demand function is given by $p = 20 - 8D + 3D^2$. Find the total revenue, average revenue and marginal revenue when the demand is 4 units.
- (4) The total revenue function is given by $R = 30D - 2D^2 + D^3$. Find the demand function. Also find total revenue, average revenue and marginal revenue when the demand is 5 units.
- (5) The demand function is given by $D = -28 - 5p + 2p^2$. Find the elasticity of demand when the price is 3.
- (6) The demand function is given by $D = \frac{2p + 5}{p - 3}$ Where $D =$ Demand and $p =$ price. Find the elasticity of demand when price is 6.
- (7) If $AR = 65$ and $\eta = 3$, find MR .
- (8) If $MR = 85$ and $\eta = 4.5$, find AR .
- (9) If $MR = 55$ and $AR = 98$, find η .
- (10) If the price is 65 and the elasticity of demand is 5.2, find the marginal revenue.

Answers:

- (1) $C = 255$; $AC = 42.5$; $MC = 112$
- (2) $C = 1280$; $AC = 128$; $MC = 345$
- (3) $R = 144$; $AR = 36$; $MR = 100$
- (4) $p = 30 - 2D + D^2$
 $R = 225$; $AR = 45$; $MR = 85$
- (5) $\eta = 0.64$
- (6) $\eta = 1.29$
- (7) $MR = 43.3$

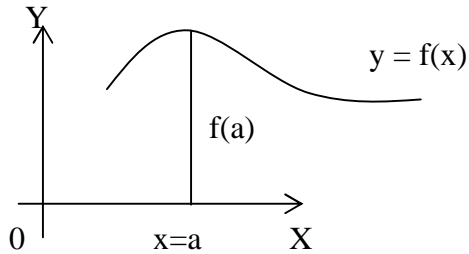
(8) $AR = 108.97$

(9) $\eta = 2.28$

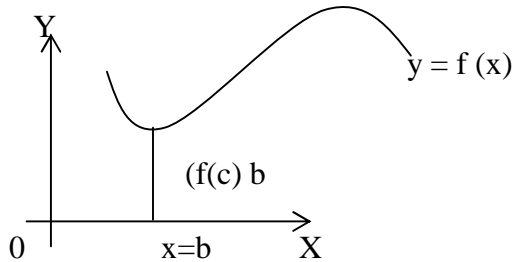
(10) $MR = 52.5$

1.5 MAXIMA AND MINIMA

Let $y = f(x)$ be the given function. A curve $f(x)$ is said to have a maximum or minimum point (extreme point), if $f(x)$ attains either a maximum or minimum of that point.



Maximum at $x = a$



Maximum at $x = b$

In the first figure, $x = a$ is the point where the curve $f(x)$ attains a maximum. In the second figure, $x = b$ is the point where the curve $f(x)$ attains a minimum

1.5.1 Conditions for Maximum & Minimum:

1. Condition for Maximum:

- (i) $f'(x) = 0$
- (ii) $f''(x) < 0$ at $x = a$

2. Condition for Minimum:

- (i) $f'(x) = 0$
- (ii) $f''(x) > 0$ at $x = b$

1.5.2. To find the Maximum and Minimum values of $f(x)$:

Steps:

- (i) Find $f'(x)$ and $f''(x)$.
- (ii) put $f'(x) = 0$, solve and get the values of x .
- (iii) Substitute the values of x in $f''(x)$.

If $f''(x) < 0$, then $f(x)$ has maximum value at $x = a$. If $f''(x) > 0$, then $f(x)$ has minimum value at $x = b$.

(iv) To find the maximum and minimum values, put the points $x = a$ and $x = b$ in $f(x)$.

Note: Extreme values of $f(x)$ = Maximum and minimum values of $f(x)$.

1.5.3. Examples :

Ex: (1) Find the extreme values of $f(x) = x^3 - 3x^2 - 45x + 25$.

Solution:

$$\text{Given : } f(x) = x^3 - 3x^2 - 45x + 25$$

$$\therefore f'(x) = 3x^2 - 6x - 45$$

$$f''(x) = 6x - 6.$$

Since $f(x)$ has maximum or minimum value,

$$f'(x) = 0$$

$$\therefore 3x^2 - 6x - 45 = 0$$

$$3(x^2 - 2x - 15) = 0$$

$$\therefore x^2 - 2x - 15 = 0$$

$$x^2 - 5x + 3x - 15 = 0$$

$$x(x - 5) + 3(x - 5) = 0$$

$$(x - 5)(x + 3) = 0$$

$$x - 5 = 0 \text{ or } x + 3 = 0$$

$$\therefore x = 5 \text{ or } x = -3.$$

When $x = 5$, $f''(5) = 6(5) - 6 = 24 > 0$

$\therefore f(x)$ has minimum at $x = 5$.

When $x = -3$, $f''(-3) = 6(-3) - 6$

$$= -24 < 0$$

$\therefore f(x)$ has maximum at $x = -3$.

To find the maximum and minimum values of $f(x)$:

put $x = 5$ and $x = -3$ in $f(x)$.

$$\therefore f(5) = 5^3 - 3(5)^2 - 45(5) + 25$$

$$= 125 - 75 - 225 + 25$$

$$= -150$$

$$f(-3) = (-3)^3 - 3(-3)^2 - 45(-3) + 25$$

$$= -27 - 27 + 135 + 25$$

$$= 106$$

\therefore Maximum value = 106 at $x = -3$

Minimum value = -150 at $x = 5$.

Ex: (2) Find the maximum and minimum values of $f(x) = x + \frac{16}{x}$, $x \neq 0$.

Solution : Given : $f(x) = x + (16/x)$

$$\therefore f'(x) = 1 + 16(-1/x^2)$$

$$f'(x) = 1 - \frac{16}{x^2}$$

$$\therefore f''(x) = 0 - 16\left(\frac{-2}{x^3}\right)$$

$$f''(x) = \frac{32}{x^3}$$

Since $f(x)$ has maximum or minimum value, $f'(x) = 0$.

$$\therefore 1 - \frac{16}{x^2} = 0$$

$$\frac{x^2 - 16}{x^2} = 0$$

$$x^2 - 16 = 0$$

$$x^2 - 4^2 = 0$$

$$(x - 4)(x + 4) = 0$$

$$x = 4 \text{ or } x = -4$$

$$\text{when } x = 4, f''(4) = \frac{32}{(4)^3} = \frac{32}{64} = \frac{1}{2} > 0$$

$\therefore f(x)$ has minimum at $x = 4$.

$$\text{when } x = -4, f''(-4) = \frac{32}{(-4)^3} = \frac{32}{-64} = -\frac{1}{2} < 0$$

$\therefore f(x)$ has maximum at $x = -4$.

Now to find the extreme values of $f(x)$: put $x = 4$ and $x = -4$ in $f(x)$

$$\therefore f(4) = 4 + \frac{16}{4} = 4 + 4 = 8$$

$$f(-4) = -4 + \frac{16}{(-4)} = -4 - 4 = -8$$

\therefore Maximum value = -8 at $x = -4$

Minimum value = 8 at $x = 4$.

Ex: (3) Divide 80 into two parts such that the sum of their squares is a minimum.

Solution:

Let x and $80 - x$ be the two required numbers.

\therefore By the given condition,

$$f(x) = x^2 + (80 - x)^2$$

$$f(x) = x^2 + 80^2 - 2(80)(x) + x^2$$

$$f(x) = 2x^2 - 160x + 6400$$

$$\therefore f'(x) = 4x - 160$$

$$f''(x) = 4$$

Since $f(x)$ has minimum,

$$f'(x) = 0$$

$$4x - 160 = 0$$

$$4x = 160$$

$$x = \frac{160}{4} = 40.$$

$$\therefore f''(x) = 4 > f''(40) = 4 > 0.$$

$\therefore f(x)$ has minimum at $x = 40$.

\therefore The required numbers are
40 and $80 - 40 = 40$

\therefore The required parts of 80 are
40 and 40.

Ex: (4)

A manufacturer can sell x items at a price of Rs. $(330 - x)$ each. The cost of producing x items is Rs. $(x^2 + 10x + 12)$. Find x for which the profit is maximum.

Solution:

Given that the total cost function is
 $C = x^2 + 10x + 12$.

Selling price $p = 330 - x$

Revenue is $R = p \times D$

$$= p \times x \quad (D = x)$$

$$= (330 - x) \times x$$

$$= 330x - x^2$$

\therefore Profit = Revenue - Cost

$\therefore P = R - C$

$$= (330x - x^2) - (x^2 + 10x + 12)$$

$$= 330x - x^2 - x^2 - 10x - 12$$

$$P = 320x - 2x^2 - 12$$

$$\therefore \frac{dp}{dx} = 320 - 4x$$

$$\frac{d^2p}{dx^2} = -4 < 0$$

\therefore The profit is maximum.

Since the profit is maximum,

$$\frac{dp}{dx} = 0$$

$$\begin{aligned}\therefore 320-4x &= 0 \\ \therefore 4x &= 320 \\ x &= 80.\end{aligned}$$

Hence the profit is maximum when 80 items are sold.

Ex: (5) The total cost function is

$$C = x^3 - 9x^2 + 24x + 70.$$

Find x for which the total cost is minimum.

Solution:

$$\text{Let } C = f(x) = x^3 - 9x^2 + 24x + 70$$

$$\therefore C' = f'(x) = 3x^2 - 18x + 24$$

$$C'' = f''(x) = 6x - 18$$

Since $f(x)$ has minimum, $f'(x) = 0$

$$\therefore 3x^2 - 18x + 24 = 0$$

$$\therefore 3(x^2 - 6x + 8) = 0$$

$$\therefore x^2 - 6x + 8 = 0$$

$$x^2 - 2x - 4x + 8 = 0$$

$$x(x-2) - 4(x-2) = 0$$

$$(x-2)(x-4) = 0$$

$$x-2=0 \text{ or } x-4=0$$

$$x=2 \text{ or } x=4$$

When $x=4$

$$f''(x) = 6x - 18$$

$$\begin{aligned}f''(4) &= 6(4) - 18 \\ &= 6 > 0\end{aligned}$$

$\therefore f(x)$ has minimum at $x=4$.

\therefore The total cost is minimum at $x=4$.

Ex: (6) The total revenue function is given by

$$R = 4x^3 - 72x^2 + 420x + 800.$$

Find x for which the total revenue is maximum.

Solution:

$$\text{Let } R = f(x) = 4x^3 - 72x^2 + 420x + 800$$

$$\therefore R' = f'(x) = 12x^2 - 144x + 420$$

$$R'' = f''(x) = 24x - 144$$

Since $f(x)$ has maximum,

$$\begin{aligned}
 f'(x) &= 0 \\
 \therefore 12x^2 - 144x + 420 &= 0 \\
 \therefore 12(x^2 - 12x + 35) &= 0 \\
 \therefore x^2 - 12x + 35 &= 0 \\
 x^2 - 5x - 7x + 35 &= 0 \\
 x(x-5) - 7(x-5) &= 0 \\
 (x-5)(x-7) &= 0 \\
 x-5=0 \text{ or } x-7=0 \\
 \therefore x=5 \text{ or } x=7
 \end{aligned}$$

When $x=5$

$$\begin{aligned}
 R'' = f''(x) &= 24x - 144 \\
 &= 24(5) - 144 \\
 &= -24 < 0
 \end{aligned}$$

$\therefore f(x)$ has maximum at $x=5$

\therefore The total revenue is maximum at $x=5$

Exercise : 1.5

(1) Find the extreme values of $f(x) = 2x^3 - 6x^2 - 48x + 90$.

(2) Find the maximum and minimum values of $f(x) = x + (9/x)$, $x \neq 0$

(3) Find the extreme values of $f(x) = 4x^3 - 12x^2 - 36x + 25$

(4) Find the extreme values of $f(x) = x + (36/x)$, $x \neq 0$.

(5) Divide 120 into two parts such that their product is maximum.

(6) Divide 70 into two parts such that the sum of their squares is a minimum.

(7) A manufacturer sells x items at a price of Rs. $(400-x)$ each. The cost of producing x items is Rs $(x^2 + 40x + 52)$. Find x for which the profit is maximum.

(8) The cost function is given by $C = x^3 - 24x^2 + 189x + 120$. Find x for which the cost is minimum,

(9) The total revenue function is given by

$$R = 2x^3 - 63x^2 + 648x + 250.$$

Find x for which the total revenue is maximum.

(10) The total cost of producing x units is Rs. $(x^2 + 2x + 5)$ and the price is Rs. $(30-x)$ per unit. Find x for which the profit is maximum.

Answers:

(1) Maximum value= 146 at $x=-2$
Minimum value = -70 at $x=4$

(2) Maximum value= -6 at $x=-3$
Minimum value = 6 at $x=3$

(3) Maximum value= 45 at $x=-1$
Minimum value= -83 at $x=3$

(4) Maximum value = -12 at $x=-6$
Minimum value = 12 at $x=6$

(5) 60,60

(6) 35,35

(7) 90

(8) 9

(9) 9

(10) 7



Unit-2

SIMPLE INTEREST AND COMPOUND INTEREST

Unit Structure :

- 2.0 Objectives
- 2.1 Introduction
- 2.2 Definitions of Terms Used In This Chapter
- 2.3 Simple Interest
- 2.4 Compound Interest

2.0 OBJECTIVES

After reading this chapter you will be able to:

- Define interest, principal, rate of interest, period.
- Find simple interest (SI), rate of S.I., period of investment.
- Find Compound Interest (CI), rate of C.I., Amount accumulated at the end of a period.
- Compound interest compounded yearly, half-yearly, quarterly or monthly.

2.1 INTRODUCTION

In every day life individuals and business firms borrow money from various sources for different reasons. This amount of money borrowed has to be returned from the lender in a stipulated time by paying some *interest* on the amount borrowed. In this chapter we are going to study the two types of interests viz. simple and compound interest. We start with some definitions and then proceed with the formula related to both the types of interests.

2.2 DEFINITIONS OF TERMS USED IN THIS CHAPTER

Principal: The sum borrowed by a person is called its *principal*. It is denoted by *P*.

Period: The time span for which money is lent is called *period*. It is denoted by *n*.

Interest: The amount paid by a borrower to the lender for the use of money borrowed for a certain period of time is called *Interest*. It is denoted by *I*.

Rate of Interest: This is the interest to be paid on the amount of Rs. 100 per annum (i.e. per year). This is denoted by *r*.

Total Amount: The sum of the principal and interest is called as the total amount (*maturity value*) and is denoted by *A*. Thus, $A = P + I$.
i.e. Interest paid $I = A - P$.

2.3 SIMPLE INTEREST

The interest which is payable on the principal only is called as *simple interest* (S.I.). For example the interest on Rs. 100 at 11% after one year is Rs.11 and the amount is $100 + 11 =$ Rs. 111.

It is calculated by the formula: $I = \frac{Pnr}{100} = P \times n \times \frac{r}{100}$

Simple Interest = Principal x period x rate of interest

$$\text{Amount at the end of } n^{\text{th}} \text{ year} = A = P + I = P + \frac{Pnr}{100} = P \left(1 + \frac{nr}{100} \right)$$

Remark: The period *n* is always taken in ‘years’. If the period is given in months/days, it has to be converted into years and used in the above formula. For example, if period is 4 months then we take $n = 4/12 = 1/3$ or if period is 60 days then $n = 60/365$.

Example 1: If Mr. Sagar borrows Rs. 500 for 2 years at 10% rate of interest, find (i) simple interest and (ii) total amount.

Ans: Given $P =$ Rs. 500, $n = 2$ and $r = 10\%$

$$(i) I = \frac{Pnr}{100} = \frac{500 \times 2 \times 10}{100} = \text{Rs. } 100$$

$$(ii) A = P + I = 500 + 100 = \text{Rs. } 600$$

3.3.1 Problems involving unknown factors in the formula $I = \frac{Pnr}{100}$

The formula $I = \frac{Pnr}{100}$ remaining the same, the unknown factor in the formula is taken to the LHS and its value is computed. For example, if rate of interest is unknown then the formula is rewritten as $r = \frac{I \times 100}{P \times n}$.

Example 2: If Mr. Prashant borrows Rs. 1000 for 5 years and pays an interest of Rs. 300, find rate of interest.

Ans: Given $P = 1000$, $n = 5$ and $I =$ Rs. 300

$$\text{Now, } I = \frac{Pnr}{100} \quad \Rightarrow r = \frac{I \times 100}{P \times n} = \frac{300 \times 100}{1000 \times 5} = 6$$

Thus, the rate of interest is 6%.

Example 3: Find the period for Rs. 2500 to yield Rs. 900 in simple interest at 12%.

Ans: Given $P = \text{Rs. } 2500, I = 900, r = 12\%$

$$\text{Now, } I = \frac{Pnr}{100} \quad \Rightarrow n = \frac{I \times 100}{P \times r} = \frac{900 \times 100}{2500 \times 12} = 3$$

Thus, the period is 3 years.

Example 4: Find the period for Rs. 1000 to yield Rs. 50 in simple interest at 10%.

Ans: Given $P = \text{Rs. } 1000, I = 50, r = 10\%$

$$\text{Now, } I = \frac{Pnr}{100} \quad \Rightarrow n = \frac{I \times 100}{P \times r} = \frac{50 \times 100}{1000 \times 10} = 0.5$$

Thus, the period is 0.5 years i.e. 6 months.

Example 5: Mr. Akash lent Rs. 5000 to Mr. Prashant and Rs. 4000 to Mr. Sagar for 5 years and received total simple interest of Rs. 4950. Find (i) the rate of interest and (ii) simple interest of each.

Ans: Let the rate of interest be r .

$$\text{S.I. for Prashant} = \frac{5000 \times 5 \times r}{100} = 250r \quad \dots (1)$$

$$\text{and S.I. for Sagar} = \frac{4000 \times 5 \times r}{100} = 200r \quad \dots (2)$$

from (1) and (2), we have,

$$\begin{aligned} \text{total interest from both} &= 250r + 200r \\ &= 450r \end{aligned}$$

But total interest received by Mr. Akash = Rs. 4950

$$\therefore 450r = 4950 \quad \Rightarrow r = \frac{4950}{450} = 11$$

\therefore the rate of interest = 11%

Example 6: The S.I. on a sum of money is one-fourth the principal. If the period is same as that of the rate of interest then find the rate of interest.

Ans: Given $I = \frac{P}{4}$ and $n = r$

$$\text{Now, we know that } I = \frac{Pnr}{100}$$

$$\therefore \frac{P}{4} = \frac{P \times r \times r}{100} \quad \Rightarrow \quad \frac{100}{4} = r^2$$

$$\therefore r^2 = 25 \quad \Rightarrow r = 5.$$

\therefore the rate of interest = 5%

Example 7: If Rs. 8400 amount to Rs. 11088 in 4 years, what will Rs. 10500 amount to in 5 years at the same rate of interest?

Ans:

(i) Given $n = 4$, $P = \text{Rs. } 8400$, $A = \text{Rs. } 11088$

$$\therefore I = A - P = 11088 - 8400 = \text{Rs. } 2688$$

Let r be the rate of interest.

$$\text{Now, } I = \frac{Pnr}{100} \quad \Rightarrow 2688 = \frac{8400 \times 4 \times r}{100}$$

$$\therefore r = 8\%$$

(ii) To find A when $n = 5$, $P = \text{Rs. } 10500$, $r = 8$

$$A = P \left(1 + \frac{nr}{100} \right) = 10500 \times \left(1 + \frac{5 \times 8}{100} \right) = 10500 \times \frac{140}{100} = 14700$$

\therefore the required amount = Rs. 14,700

Example 8: Mr. Shirish borrowed Rs. 12,000 at 9% interest from Mr. Girish on January 25, 2007. The interest and principal is due on August 10, 2007. Find the interest and total amount paid by Mr. Shirish.

Ans: Since the period is to be taken in years, we first count number of days from 25th January to 10th August, which is **197** days.

$$\text{Now, } I = \frac{Pnr}{100} = 12000 \times \frac{197}{365} \times \frac{9}{100}$$

$$\therefore I = \text{Rs. } 582.9$$

$$\text{Total amount} = P + I = 12000 + 582.9$$

$$\therefore A = \text{Rs. } 12,582.9$$

January	6
February	28
March	31
April	30
May	31
June	30
July	31
August	10
Total	197

Check your progress 10.1

1. Find the SI and amount for the following data giving principal, rate of interest and number of years:

(i) 1800, 6%, 4 years.

(ii) 4500, 8%, 5 years

(iii) 7650, 5.5%, 3 years.

(iv) 6000, 7.5%, 6 years

(v) 25000, 8%, 5 years

(vi) 20000, 9.5%, 10 years.

Ans: (i) 432, 2232

(ii) 1800, 6300,

(iii) 1262.25, 8912.25

(iv) 2700, 8700

(v) 10000, 35000

(vi) 19000, 39000

2. Find the S.I. and the total amount for a principal of Rs. 6000 for 3 years at 6% rate of interest.

Ans: 1080, 7080

3. Find the S.I. and the total amount for a principal of Rs. 3300 for 6 years at $3\frac{1}{2}$ % rate of interest.

Ans: 693, 3993

4. Find the S.I. and the total amount for a principal of Rs. 10550 for 2 years at $10\frac{1}{4}$ % rate of interest.
Ans: 2162.75, 12712.75
5. Find the rate of interest if a person invests Rs. 1000 for 3 years and receives a S.I. of Rs. 150.
Ans: 5%
6. Find the rate of interest if a person invests Rs. 1200 for 2 years and receives a S.I. of Rs. 168.
Ans: 7%
7. A person invests Rs. 4050 in a bank which pays 7% S.I. What is the balance of amount of his savings after (i) six months, (ii) one year?
Ans: 141.75, 283.5
8. A person invests Rs. 3000 in a bank which offers 9% S.I. After how many years will his balance of amount will be Rs. 3135?
Ans: 6 months
9. Find the principal for which the SI for 4 years at 8% is 585 less than the SI for $3\frac{1}{2}$ years at 11%.
Ans: 9000
10. Find the principal for which the SI for 5 years at 7% is 250 less than the SI for 4 years at 10%.
Ans: 5000
11. Find the principal for which the SI for 8 years at 7.5% is 825 less than the SI for $6\frac{1}{2}$ years at 10.5%.
Ans: 10000
12. Find the principal for which the SI for 3 years at 6% is 230 more than the SI for $3\frac{1}{2}$ years at 5%.
Ans: 46000
13. After what period of investment would a principal of Rs. 12,350 amount to Rs. 17,043 at 9.5% rate of interest?
Ans: 4 years
14. A person lent Rs. 4000 to Mr. X and Rs. 6000 to Mr. Y for a period of 10 years and received total of Rs. 3500 as S.I. Find (i) rate of interest, (ii) S.I. from Mr. X, Mr. Y.
Ans: 3.5%, 1400, 2100
15. Miss Pankaj Kansra lent Rs. 2560 to Mr. Abhishek and Rs. 3650 to Mr. Ashwin at 6% rate of interest. After how many years should he receive a total interest of Rs. 3726?
Ans: 10 years

16. If the rate of S.I. on a certain principal is same as that of the period of investment yields same interest as that of the principal, find the rate of interest.

Ans: 10%

17. If the rate of S.I. on a certain principal is same as that of the period of investment yields interest equal to one-ninth of the principal, find the rate of interest.

Ans: $3\frac{1}{3}$ years

18. Find the principal and rate of interest if a certain principal amounts to Rs. 2250 in 1 year and to Rs. 3750 in 3 years.

Ans: 1500, 50%

19. Find the principal and rate of interest if a certain principal amounts to Rs. 3340 in 2 years and to Rs. 4175 in 3 years. **Ans:** 1670, 50%

20. If Rs. 2700 amount Rs. 3078 in 2 years at a certain rate of interest, what will Rs. 7200 amount to in 4 years at the same rate on interest?

Ans: 7%, 9216

21. At what rate on interest will certain sum of money amount to three times the principal in 20 years?

Ans: 15%

22. Mr. Chintan earns as interest Rs. 1020 after 3 years by lending Rs. 3000 to Mr. Bhavesh at a certain rate on interest and Rs. 2000 to Mr. Pratik at a rate on interest 2% more than that of Mr. Bhavesh. Find the rates on interest.

Ans: 6%, 8%

23. Mr. Chaitanya invested a certain principal for 3 years at 8% and received an interest of Rs. 2640. Mr. Chihar also invested the same amount for 6 years at 6%. Find the principal of Mr. Chaitanya and the interest received by Mr. Chihar after 6 years.

Ans: 11000, 3960

24. Mr. Ashfaque Khan invested some amount in a bank giving 8.5% rate of interest for 5 years and some amount in another bank at 9% for 4 years. Find the amounts invested in both the banks if his total investment was Rs. 75,000 and his total interest was Rs. 29,925.

Ans: 45000, 30000

25. Mrs. Prabhu lent a total of Rs. 48,000 to Mr. Diwakar at 9.5% for 5 years and to Mr. Ratnakar at 9% for 7 years. If she receives a total interest of Rs. 25,590 find the amount she lent to both.

Ans: 18000, 30000

2.4 COMPOUND INTEREST

The interest which is calculated on the amount in the previous year is called *compound interest*.

For example, the compound interest on Rs. 100 at 8% after one year is Rs. 8 and after two years is 108 + 8% of 108 = Rs. 116.64

If P is the principal, r is the rate of interest p.a. then the amount at the end of n^{th} year called as *compound amount* is given by the formula:

$$A = P \left(1 + \frac{r}{100} \right)^n$$

The *compound interest* is given by the formula:

$$CI = A - P$$

Note:

1. The interest may be compounded annually (yearly), semi-annually (half yearly), quarterly or monthly. Thus, the general formula to calculate the amount at the end of n years is as follows:

$$A = P \left(1 + \frac{r}{p \times 100} \right)^{np}$$

Here p : number of times the interest is compounded in a year.

$p = 1$ if interest is compounded **annually**

$p = 2$ if interest is compounded **semi-annually (half-yearly)**

$p = 4$ if interest is compounded **quarterly**

$p = 12$ if interest is compounded **monthly**

2. It is easy to calculate amount first and then the compound interest as compared with finding interest first and then the total amount in case of simple interest.

Example 9: Find the compound amount and compound interest of Rs. 1000 invested for 10 years at 8% if the interest is compounded annually.

Ans: Given $P = 1000$, $r = 8$, $n = 10$.

Since the interest is compounded annually, we have

$$A = P \left(1 + \frac{r}{100} \right)^n = 1000 \times \left(1 + \frac{8}{100} \right)^{10} = 1000 \times 2.1589 = \text{Rs. } 2158.9$$

Example 10: Find the principal which will amount to Rs. 11,236 in 2 years at 6% compound interest compounded annually.

Ans: Given $A = \text{Rs. } 11236$, $n = 2$, $r = 6$ and $P = ?$

$$\text{Now, } A = P \left(1 + \frac{r}{100} \right)^n$$

$$\therefore 11236 = P \left(1 + \frac{6}{100} \right)^2 = P \times 1.1236$$

$$\therefore P = \frac{11236}{1.1236} = 10,000$$

\therefore the required principal is Rs. 10,000.

Example 11: Find the compound amount and compound interest of Rs. 1200 invested for 5 years at 5% if the interest is compounded (i) annually, (ii) semi annually, (iii) quarterly and (iv) monthly.

Ans: Given $P = \text{Rs. } 1200$, $r = 5$, $n = 5$

$$\text{Let us recollect the formula } A = P \left(1 + \frac{r}{p \times 100} \right)^{np}$$

(i) If the interest is compounded annually, $p = 1$:

$$A = P \left(1 + \frac{r}{100} \right)^n = 1200 \times \left(1 + \frac{5}{100} \right)^5 = 1200 \times 1.2763 = \text{Rs. } 1531.56$$

$$CI = A - P = 1531.56 - 1200 = \text{Rs. } 331.56$$

(ii) If the interest is compounded semi-annually, $p = 2$:

$$A = P \left(1 + \frac{r}{2 \times 100} \right)^{2n} = 1200 \times \left(1 + \frac{5}{200} \right)^{10} = 1200 \times 1.28 = \text{Rs. } 1536$$

$$CI = A - P = 1536 - 1200 = \text{Rs. } 336.$$

(iii) If the interest is compounded quarterly, $p = 4$:

$$A = P \left(1 + \frac{r}{4 \times 100} \right)^{4n} = 1200 \times \left(1 + \frac{5}{400} \right)^{20} = 1200 \times 1.2820 =$$

Rs. 1538.4

$$CI = A - P = 1538.4 - 1200 = \text{Rs. } 338.4$$

(iv) If the interest is compounded monthly, $p = 12$:

$$A = P \left(1 + \frac{r}{12 \times 100} \right)^{12n} = 1200 \times \left(1 + \frac{5}{1200} \right)^{60} = 1200 \times 1.2834 = \text{Rs. } 1540$$

$$CI = A - P = 1540 - 1200 = \text{Rs. } 340$$

Example 12: Mr. Santosh wants to invest some amount for 4 years in a bank. Bank X offers 8% interest if compounded half yearly while bank Y offers 6% interest if compounded monthly. Which bank should Mr. Santosh select for better benefits?

Ans: Given $n = 4$.

Let the principal Mr. Santosh wants to invest be $P = \text{Rs. } 100$

From Bank X: $r = 8$ and interest is compounded half-yearly, so $p = 2$.

$$\therefore A = P \left(1 + \frac{r}{2 \times 100} \right)^{2n} = 100 \times \left(1 + \frac{8}{200} \right)^4 = 116.9858 \quad \dots (1)$$

From Bank Y: $r = 6, p = 12$

$$\therefore A = P \left(1 + \frac{r}{12 \times 100} \right)^{12n} = 100 \times \left(1 + \frac{6}{1200} \right)^{48} = 127.0489 \quad \dots (2)$$

Comparing (1) and (2), Dr. Ashwinikumar should invest his amount in bank Y as it gives more interest at the end of the period.

Example 13: In how many years would Rs. 75,000 amount to Rs. 1,05,794.907 at 7% compound interest compounded semi-annually?

Ans: Given $A = \text{Rs. } 105794.907$, $P = \text{Rs. } 75000$, $r = 7$, $p = 2$

$$A = P \left(1 + \frac{r}{2 \times 100} \right)^{2n}$$

$$\therefore 105794.907 = 75000 \times \left(1 + \frac{7}{200} \right)^{2n}$$

$$\therefore \frac{105794.907}{75000} = (1.035)^{2n}$$

$$\therefore 1.41059876 = (1.035)^{2n}$$

$$\therefore (1.035)^{10} = (1.035)^{2n} \quad \Rightarrow 2n = 10$$

Thus, $n = 5$

Example 14: A certain principal amounts to Rs. 4410 after 2 years and to Rs. 4630.50 after 3 years at a certain rate of interest compounded annually. Find the principal and the rate of interest.

Ans: Let the principal be P and rate of interest be r .

$$\text{Now, we know that } A = P \left(1 + \frac{r}{100} \right)^n$$

From the given data we have,

$$4410 = P \left(1 + \frac{r}{100} \right)^2 \quad \text{and} \quad 4630.5 = P \left(1 + \frac{r}{100} \right)^3$$

$$\therefore 4410 = P(1 + 0.01r)^2 \quad \dots (1)$$

$$4630.5 = P(1 + 0.01r)^3 \quad \dots (2)$$

Do not write '1 + 0.01r'
as 1.01r

Dividing (2) by (1), we have

$$\frac{4630.5}{4410} = \frac{P(1+0.01r)^3}{P(1+0.01r)^2} \Rightarrow 1.05 = 1 + 0.01r$$

$$\therefore 0.05 = 0.01r$$

Thus, $r = 5\%$

Example 15: Find the rate of interest at which a sum of Rs. 2000 amounts to Rs. 2690 in 3 years given that the interest is compounded half yearly.

$$(\sqrt[6]{1.345} = 1.05)$$

Ans: Given $P = \text{Rs. } 2000$, $A = \text{Rs. } 2680$, $n = 3$, $p = 2$

$$\text{Now, } A = P \left(1 + \frac{r}{2 \times 100} \right)^{2n}$$

$$\therefore 2690 = 2000 \times \left(1 + \frac{r}{200} \right)^6$$

$$\therefore \frac{2690}{2000} = \left(1 + \frac{r}{200} \right)^6 \Rightarrow 1.345 = \left(1 + \frac{r}{200} \right)^6$$

$$\therefore \sqrt[6]{1.345} = 1 + \frac{r}{200} \quad \Rightarrow 1.05 = 1 + \frac{r}{200}$$

$$\therefore r = 0.05 \times 200 = 10\%$$

Thus, the rate of compound interest is **10 %**.

Example 16: If the interest compounded half yearly on a certain principal at the end of one year at 8% is Rs. 3264, find the principal.

Ans: Given $CI = \text{Rs. } 3264$, $n = 1$, $p = 2$ and $r = 8$

$$\text{Now, } CI = A - P = P \left(1 + \frac{8}{200} \right)^2 - P$$

$$\text{i.e. } 3264 = P[(1.04)^2 - 1] = 0.0816P$$

$$\therefore P = \frac{3264}{0.0816} = 40000$$

Thus, the principal is **Rs. 40,000**.

Check your progress 10.2

1. Compute the compound amount and interest on a principal of Rs. 21,000 at 9% p.a. after 5 years.

Ans: 32,311.10, 11,311.10

2. Compute the compound amount and interest on a principal of Rs. 6000 at 11% p.a. after 8 years.

Ans: 13827.23, 7827.23

3. Compute the compound amount and compound interest of Rs. 5000 if invested at 11% for 3 years and the interest compounded *i*) annually, *ii*) semi annually, *iii*) quarterly and *iv*) monthly.

Ans: *i*) 6838.16, 1838.16 *ii*) 6894.21, 1894.21

iii) 6923.92, 1923.92 *iv*) 6944.39, 1944.39

4. Compute the compound amount and compound interest of Rs. 1200 if invested at 9% for 2 years and the interest compounded *i*) annually, *ii*) semi annually, *iii*) quarterly and *iv*) monthly.
Ans: *i*) 1425.72, 225.72 *ii*) 1431.02, 231.02
iii) 1433.8, 233.8 *iv*) 1435.7, 235.7
5. Miss Daizy invested Rs. 25,000 for 5 years at 7.5% with the interest compounded semi-annually. Find the compound interest at the end of 5 years. **Ans:**
 11,126.10
6. Mr. Dayanand borrowed a sum of Rs. 6500 from his friend at 9% interest compounded quarterly. Find the interest he has to pay at the end of 4 years? **Ans:** 2779.54
7. Mr. Deepak borrowed a sum of Rs. 8000 from his friend at 8% interest compounded annually. Find the interest he has to pay at the end of 3 years? **Ans:**
 2077.70
8. Mr. Deshraj borrowed Rs. 1,25,000 for his business for 3 years at 25% interest compounded half yearly. Find the compound amount and interest after 3 years. **Ans:** 2,53,410.82;
 12,8410.82
9. Mrs. Hemlata bought a Sony Digital Camera for Rs. 15,800 from Vijay Electronics by paying a part payment of Rs. 2,800. The remaining amount was to be paid in 3 months with an interest of 9% compounded monthly on the due amount. How much amount did Mrs. Hemlata paid and also find the interest. **Ans:** 13294.70, 294.70
10. Mr. Irshad bought a Kisan Vikas Patra for Rs. 10000, whose maturing value is Rs. 21,000 in 4½ years. Calculate the rate of interest if the compound interest is compounded quarterly. **Ans:** 16.8%
11. What sum of money will amount to Rs. 11236 in 2 years at 6% p.a. compound interest? **Ans:** 10,000
12. Find the principal which will amount to Rs. 13468.55 in 5 years at 6% interest compounded quarterly. [$(1.015)^{20} = 1.346855$] **Ans:** 10000
13. Find the principal which will amount to Rs. 30626.075 in 3 years at 7% interest compounded yearly. **Ans:** 25000
14. Find the principal if the compound interest payable annually at 8% p.a. for 2 years is Rs. 1664. **Ans:** 10000

15. If Mr. Sagar wants to earn Rs. 50000 after 4 years by investing a certain amount in a company at 10% rate of interest compounded annually, how much should he invest? **Ans:**
34150.67
16. Find after how many years will Rs. 4000 amount to Rs. 4494.40 at 6% rate of interest compounded yearly. **Ans:** $n = 2$
17. Find after how many years Rs. 10,000 amount to Rs. 12,155 at 10% rate of interest compounded half-yearly. **Ans:** $n = 1$
18. Find the rate of interest at which a principal of Rs.10000 amounts to Rs. 11236 after 2 years. **Ans:** 6%
19. Find the rate of interest at which a principal of Rs.50000 amounts to Rs. 62985.6 after 3 years. ($\sqrt[3]{1.259712} = 1.08$) **Ans:** 8%
20. Mrs. Manisha Lokhande deposited Rs. 20,000 in a bank for 5 years. If she received Rs.3112.50 as interest at the end of 2 years, find the rate of interest p.a. compounded annually. **Ans:** 7.5%
21. A bank X announces a super fixed deposit scheme for its customers offering 10% interest compounded half yearly for 6 years. Another bank Y offers 12% simple interest for the same period. Which bank's scheme is more beneficial for the customers? **Ans:** Bank X
22. ABC bank offers 9% interest compounded yearly while XYZ bank offers 7% interest compounded quarterly. If Mr. Arunachalam wants to invest Rs. 18000 for 5 years, which bank should he choose? **Ans:** Bank ABC
23. Mangesh borrowed a certain amount from Manish at a rate of 9% for 4 years. He paid Rs. 360 as simple interest to Manish. This amount he invested in a bank for 3 years at 11% rate of interest compounded quarterly. Find the compound interest Mangesh received from the bank after 3 years. **Ans:**
1384.78
24. On a certain principal for 3 years the compound interest compounded annually is Rs. 1125.215 while the simple interest is Rs. 1050, find the principal and the rate of interest. **Ans:** 5000, 7%
25. On a certain principal for 4 years the compound interest compounded annually is Rs. 13923 while the simple interest is Rs. 12000, find the principal and the rate of interest. **Ans:** 30000, 10%.

26. Which investment is better for Mr. Hariom Sharma (i) 6% compounded half yearly or (ii) 6.2% compounded quarterly?

Ans:

27. Which investment is better for Mr. Suyog Apte (i) 9% compounded yearly or (ii) 8.8% compounded quarterly?

Ans:

28. A bank X offers 7% interest compounded semi-annually while another bank offers 7.2% interest compounded monthly. Which bank gives more interest at the end of the year?

Ans:

29. Mr. Nitin Tare has Rs. 10000 to be deposited in a bank. One bank offers 8% interest p.a. compounded half yearly, while the other offers 9% p.a. compounded annually. Calculate the returns in both banks after 3 years. Which bank offers maximum return after 3 years?

Ans:



Unit-3

ANNUITIES AND EMI

Unit Structure :

- 3.0 Objectives
- 3.1 Introduction
- 3.2 Annuity
- 3.3 Types of Annuities
- 3.4 Sinking Fund
- 3.5 Equated Monthly Installment (Emi)

3.0 OBJECTIVES

After reading this chapter you will be able to:

- Define annuity, future value, present value, EMI, Sinking Fund.
- Compute Future Value of annuity due, Present Value of an ordinary annuity.
- Compute EMI of a loan using reducing balance method and flat interest method.
- Compute Sinking Fund (periodic payments).

3.1 INTRODUCTION

In the previous chapter we have seen how to compute compound interest when a lump sum amount is invested at the beginning of the investment. But many a time we pay (or are paid) a certain amount not in lump sum but in periodic installments. This series of equal payments done at periodic intervals is called as *annuity*.

Let us start the chapter with the definition of an *annuity*.

3.2 ANNUITY

A series of equal periodic payments is called *annuity*. The payments are of *equal size* and *at equal time interval*.

The common examples of annuity are: monthly recurring deposit schemes, premiums of insurance policies, loan installments, pension installments etc. Let us understand the terms related to annuities and then begin with the chapter.

Periodic Payment:

The amount of payment made is called as *periodic payment*.

Period of Payment:

The time interval between two payments of an annuity is called as the *period of payment*.

Term of an annuity:

The time of the beginning of the first payment period to the end of the last payment period is called as *term of annuity*. An annuity gets *matured* at the end of its term.

3.3 TYPES OF ANNUITIES

Though we will be discussing two types of annuities in detail, let us understand different types of annuities based on the duration of the term or on the time when the periodic payments are made. On the basis of the closing of an annuity, there are three types of annuities:

1. Certain Annuity:

Here the duration of the annuity is fixed (or certain), hence called *certain annuity*. We will be learning such annuities in detail.

2. Perpetual Annuity:

Here the annuity has no closing duration, i.e. it has indefinite duration. Practically there are rarely any perpetuities.

3. Contingent Annuity:

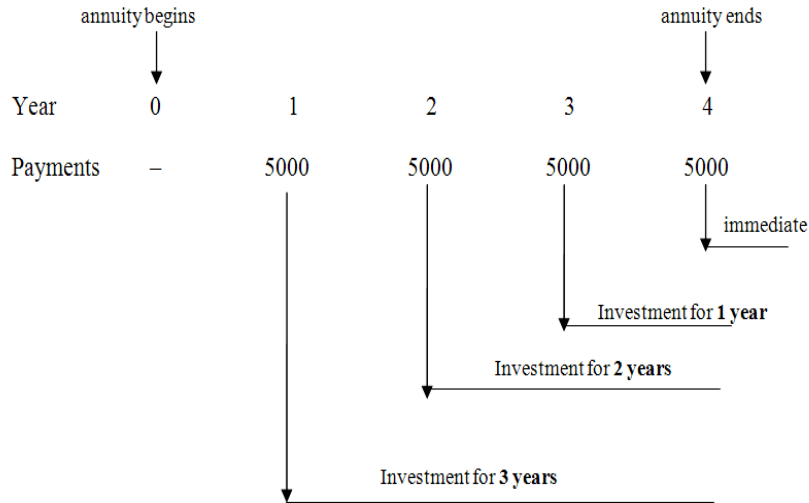
Here the duration of the annuity depends on an event taking place. An example of contingent annuity is *life annuity*. Here the payments are to be done till a person is alive, like, pension, life insurance policies for children (maturing on the child turning 18 years) etc.

On the basis of when the periodic payments are made we have two types of annuities: ordinary annuity and annuity due.

3.3.1 Immediate (Ordinary) Annuity:

The annuity which is paid at the *end of each period* is called as *immediate (ordinary) annuity*. The period can be monthly, quarterly or yearly etc. For example, stock dividends, salaries etc.

Let us consider an example of an investment of Rs. 5000 each year is to be made for four years. If the investment is done at the end of each year then we have the following diagrammatic explanation for it:



3.3.2 Present Value:

The sum of all periodic payments of an annuity is called its *present value*. In simple words, it is that sum which if paid *now* will give the same amount which the periodic payments would have given at the end of the decided period. It is the one time payment of an annuity.

The formula to find the present value (*PV*) is as follows:

$$PV = \frac{P}{\left(\frac{r}{p \times 100}\right)} \left[1 - \frac{1}{\left(1 + \frac{r}{p \times 100}\right)^{np}} \right]$$

Where

P: periodic equal payment

r: rate of interest p.a.

p: period of annuity

Let $i = \frac{r}{p \times 100}$, the rate per period, then the above formula can be

rewritten as follows:

$$PV = \frac{P}{i} \left[1 - \frac{1}{(1+i)^{np}} \right]$$

3.3.3 Future Value (Accumulated value):

The sum of all periodic payments along with the interest is called the *future value (accumulated amount)* of the annuity.

The formula to find the future value (*A*) of an immediate annuity is as follows:

$$A = \frac{P}{\frac{r}{p \times 100}} \left[\left(1 + \frac{r}{p \times 100} \right)^{np} - 1 \right]$$

$$A = \frac{P}{i} \left[(1+i)^{np} - 1 \right]$$

Here,

P : periodic equal payment

r : rate of interest p.a.

p : period of annuity i.e. yearly, half yearly, quarterly or monthly

and $i = \frac{r}{p \times 100}$

Example 1: Find the future value after 2 years of an immediate annuity of Rs. 5000, the rate of interest being 6% p.a compounded annually.

Ans: Given $n = 2$, $P = \text{Rs. } 5000$, $r = 6$ and $p = 1 \Rightarrow i = \frac{6}{100} = 0.06$

$$A = \frac{P}{i} \left[(1+i)^{np} - 1 \right] = \frac{5000}{0.06} \left[(1+0.06)^2 - 1 \right] = 5000 \left[\frac{1.1236 - 1}{0.06} \right]$$

$\therefore A = 5000 \times 2.06 = \text{Rs. } 10300$

Example 2: Find the amount for an ordinary annuity with periodic payment of Rs. 3000, at 9% p.a. compounded semi-annually for 4 years.

Ans: Given $n = 4$, $P = \text{Rs. } 3000$, $r = 9$ and $p = 2 \Rightarrow i = \frac{9}{2 \times 100} = 0.045$

$$\text{Now, } A = \frac{P}{i} \left[(1+i)^{np} - 1 \right] = \frac{3000}{0.045} \left[(1+0.045)^{2 \times 4} - 1 \right] = \frac{3000}{0.045} \times 0.4221$$

Thus, $A = \text{Rs. } 28,140$

Example 3: Mr. Ravi invested Rs. 5000 in an annuity with quarterly payments for a period of 2 years at the rate of interest of 10%. Find the accumulated value of the annuity at the end of 2nd year.

Ans: Given $n = 2$, $P = \text{Rs. } 5000$, $r = 10$ and $p = 4 \Rightarrow i = \frac{10}{4 \times 100} = 0.025$

$$\text{Now, } A = \frac{P}{i} \left[(1+i)^{np} - 1 \right] = \frac{5000}{0.025} \left[(1.025)^{2 \times 4} - 1 \right] = \frac{5000}{0.025} \times 0.2184$$

Thus, $A = \text{Rs. } 43,680$

Example 4: Mr. Ashok Rane borrowed Rs. 20,000 at 4% p.a. compounded annually for 10 years. Find the periodic payment he has to make.

Ans: Given $PV = \text{Rs. } 20,000$; $n = 10$; $p = 1$ and $r = 4 \Rightarrow i = 0.04$

Now to find the periodic payments P we use the following formula:

$$PV = \frac{P}{i} \left[1 - \frac{1}{(1+i)^{np}} \right]$$

$$\therefore 20000 = \frac{P}{0.04} \left[1 - \frac{1}{(1+0.04)^{10}} \right] = \frac{P}{0.04} \times 0.3244$$

$$\therefore P = \frac{20000 \times 0.04}{0.3244} = 2466.09$$

Thus, Mr. Rane has to make the periodic payments of Rs. 2466.09

Example 5: Find the future value of an immediate annuity after 3 years with the periodic payment of Rs. 12000 at 5% p.a. if the period of payments is (i) yearly, (ii) half-yearly, (iii) quarterly and (iv) monthly.

Ans: Given $P = \text{Rs. } 1200$, $n = 3$, $r = 5$

(i) period $p = 1$ then $i = \frac{5}{100} = 0.05$

$$A = \frac{P}{i} \left[(1+i)^n - 1 \right] = \frac{12000}{0.05} \left[(1+0.05)^3 - 1 \right] = \frac{12000}{0.05} [1.1576 - 1]$$

$$\therefore A = 12000 \times 3.1525 = \text{Rs. } 37,830$$

(ii) period $p = 2$ then $i = \frac{5}{2 \times 100} = 0.025$

$$A = \frac{P}{i} \left[(1+i)^{2n} - 1 \right] = \frac{12000}{0.025} \left[(1+0.025)^6 - 1 \right] = \frac{12000}{0.025} \times 0.1597$$

$$\therefore A = 12000 \times 6.388 = \text{Rs. } 76,656$$

(iii) period $p = 4$ then $i = \frac{5}{4 \times 100} = 0.0125$

$$A = \frac{P}{i} \left[(1+i)^{4n} - 1 \right] = \frac{12000}{0.0125} \left[(1+0.0125)^{12} - 1 \right] = \frac{12000}{0.0125} \times 0.16075$$

$$\therefore A = 12000 \times 12.86 = \text{Rs. } 1,54,320$$

(iv) period $p = 12$ then $i = \frac{5}{12 \times 100} = 0.00417$

$$A = \frac{P}{i} \left[(1+i)^{12n} - 1 \right] = \frac{12000}{0.00417} \left[(1+0.00417)^{36} - 1 \right] = \frac{12000}{0.00417} \times 0.1615$$

$$\therefore A = 1200 \times 38.729 = \text{Rs. } 4,64,748$$

Example 6: Mr. Nagori invested certain principal for 3 years at 8% interest compounded half yearly. If he received Rs.72957.5 at the end of 3rd year, find the periodic payment he made. $[(1.04)^6 = 1.2653]$

Ans: Given $n = 3$, $r = 8$, $p = 2 \Rightarrow i = \frac{8}{2 \times 100} = 0.04$

$$\text{Now, } A = \frac{P}{i} \left[(1+i)^{np} - 1 \right]$$

$$\therefore 72957.5 = \frac{P}{0.04} \left[(1+0.04)^6 - 1 \right] = \frac{P}{0.04} \times 0.2653$$

$$\therefore 72957.5 = P[6.6325]$$

$$\therefore P = \frac{72957.5}{6.6325} = 11000$$

Thus, the periodic payment is **Rs. 11,000**

3.4 SINKING FUND

The fund (money) which is kept aside to accumulate a certain sum in a fixed period through periodic equal payments is called as **sinking fund**.

We can consider an example of a machine in a factory which needs to be replaced after say 10 years. The amount for buying a new machine 10 years from now may be very large, so a proportionate amount is accumulated every year so that it amounts to the required sum in 10 years. This annual amount is called as *sinking fund*. Another common example is of the *maintenance tax* collected by any Society from its members.

A sinking fund being same as an annuity, the formula to compute the terms is same as that we have learnt in section 2.3.3

Example 7: A company sets aside a sum of Rs. 15,000 annually to enable it to pay off a debenture issue of Rs. 1,80,000 at the end of 10 years. Assuming that the sum accumulates at 6% p.a., find the surplus after paying off the debenture stock.

Ans: Given $P = \text{Rs. } 15000$, $n = 10$, $r = 6 \Rightarrow i = 0.06$

$$\therefore A = \frac{P}{i} \left[(1+i)^n - 1 \right] = \frac{15000}{0.06} \times [(1 + 0.06)^{10} - 1] = \frac{15000}{0.06} \times 0.7908$$

$$\therefore A = \text{Rs. } 1,97,700$$

Thus, the surplus amount after paying off the debenture stock is
 $= 197712 - 180000 = \text{Rs. } 17712$.

Example 8: Shriniketan Co-op Hsg. Society has 8 members and collects Rs. 2500 as maintenance charges from every member per year. The rate of compound interest is 8% p.a. If after 4 years the society needs to do a work worth Rs. 100000, are the annual charges enough to bear the cost?

Ans: Since we want to verify whether Rs. 2500 yearly charges are enough or not we assume it to be P and find its value using the formula:

$$A = \frac{P}{i} \left[(1+i)^n - 1 \right]$$

Here $A = \text{Rs. } 100000$, $n = 4$, $r = 8 \Rightarrow i = 0.08$

$$\therefore P = \frac{A \times i}{(1+i)^n - 1} = \frac{100000 \times 0.08}{(1+0.08)^4 - 1} = 22192$$

Thus, the annual payment of all the members i.e. 8 members should be Rs. 22192.

$$\therefore \text{the annual payment per member} = \frac{22192}{8} = \text{Rs. } 2774$$

This payment is less than Rs. 2500 which the society has decided to take presently. Thus, the society should increase the annual sinking fund.

3.5 EQUATED MONTHLY INSTALLMENT (EMI)

Suppose a person takes a loan from a bank at a certain rate of interest for a fixed period. The equal payments which the person has to make to the bank per month are called as *equated monthly installments* in short EMI.

Thus, EMI is a kind of annuity with **period of payment being monthly** and the **present value being the sum borrowed**.

We will now study the method of finding the EMI using **reducing balance method** and **flat interest method**.

(a) Reducing balance method:

Let us recall the formula of finding the present value of an annuity.

$$PV = \frac{P}{i} \left[1 - \frac{1}{(1+i)^{np}} \right]$$

The equal periodic payment (P) is our EMI which is denoted it by E .

The present value (PV) is same as the sum (S) borrowed.

Also the period being monthly $p = 12$, $i = \frac{r}{1200}$ as we are interested in

finding **monthly** installments and n is period in **years**.

Substituting this in the above formula we have:

$$S = \frac{E}{i} \left[1 - \frac{1}{(1+i)^{12n}} \right]$$

Thus, if S is the sum borrowed for n years with rate of interest r % p.a. then the *EMI* is calculated by the formula:

$$E = \frac{S \times i}{1 - \frac{1}{(1+i)^{12n}}}$$

(b) Flat Interest Method:

Here the amount is calculated using Simple Interest for the period and the EMI is computed by dividing the amount by total number of monthly installments.

Let S denote the sum borrowed, r denote the rate of interest and n denote the duration in years, then as we know the amount using simple interest formula is $A = S\left(1 + \frac{nr}{100}\right)$. The total number of monthly installments for duration of n years is $12n$. Hence the *EMI* is calculated as

$$E = \frac{A}{12n}$$

Example 9: Mr. Sudhir Joshi has taken a loan of Rs. 10,00,000 from a bank for 10 years at 11% p.a. Find his EMI using (a) reducing balance method and (b) Flat interest method.

Ans: Given $S = \text{Rs. } 1000000$, $n = 10$, $r = 11 \Rightarrow i = \frac{11}{1200} = 0.0092$

(a) Using flat interest method:

$$A = S\left(1 + \frac{nr}{100}\right) = 1000000\left(1 + \frac{110}{100}\right) = 2100000$$

$$\text{Thus, } E = \frac{A}{12n} = \frac{2100000}{120} = 17,500 \quad \dots (1)$$

(b) Using reducing balance method:

$$\text{Now, } E = \frac{S \times i}{1 - \frac{1}{(1+i)^{12n}}} = \frac{1000000 \times 0.0092}{1 - \frac{1}{(1+0.0092)^{120}}} = 13797.65$$

$$\therefore E = \text{Rs. } 13,798 \text{ approximately} \quad \dots (2)$$

Comparing (1) and (2), we can see that the *EMI* using flat interest method is higher than by reducing balance method.

Example 10: Mr. Prabhakar Naik has borrowed a sum of Rs. 60,000 from a person at 6% p.a. and is due to return it back in 4 monthly installments. Find the *EMI* he has to pay and also prepare the amortization table of repayment.

Ans: Given $S = \text{Rs. } 60,000$; $n = 4$ months;

$$r = 6\% \Rightarrow i = \frac{6}{1200} = 0.005$$

$$\text{Now, } E = \frac{S \times i}{1 - \frac{1}{(1+i)^n}} = \frac{60000 \times 0.005}{1 - \frac{1}{(1+0.005)^4}} = \frac{300}{0.01975}$$

$$\therefore E = \text{Rs. } 15,187.97$$

Now, we will prepare the *amortization table* i.e. the table of repayment of the sum borrowed using reducing balance method.

In the beginning of the 1st month the outstanding principal is the sum borrowed i.e. Rs. 60000 and the EMI paid is Rs. 15187.97

The interest on the outstanding principal is $0.005 \times 60000 = \text{Rs. } 300 \dots (1)$

Thus, the principal repayment is $15187.97 - 300 = \text{Rs. } 14887.97 \dots (2)$

The outstanding principal (O/P) in the beginning of the 2nd month is now $60000 - 14887.97 = 45112.03$.

Note:

- (1) is called the *interest part* of the EMI and (2) is called as the *principal part* of the EMI.
- As the tenure increases the interest part reduces and the principal part increases.

This calculation can be tabulated as follows:

Month	O/P	EMI	Interest Part	Principal Part
	(a)	(b)	(c) = (a) x i	(b) - (c)
1	60000	15187.97	300	14887.97
2	45112.03	15187.97	225.56	14962.45
3	30141.02	15187.97	150.75	15037.22
4	15111.80	15187.97	75.56	15112.41

In the beginning of the 4th month the outstanding principal is Rs. 15111.80 but the actual principal repayment in that month is Rs. 15112.41. This difference is due to rounding off the values to two decimals, which leads the borrower to pay 61 paise more!!

Example 11: Mr. Shyam Rane has borrowed a sum of Rs. 100000 from a bank at 12% p.a. and is due to return it back in 5 monthly installments. Find the *EMI* he has to pay and also prepare the amortization table of repayment.

Ans: Given $S = \text{Rs. } 100000$; $n = 5$ months;

$$r = 12\% \text{ p.a.} = \frac{12}{12} = 1\% \text{ p.m} \quad \Rightarrow i = 0.01$$

$$\text{Now, } E = \frac{S \times i}{1 - \frac{1}{(1+i)^n}} = \frac{100000 \times 0.01}{1 - \frac{1}{(1+0.01)^5}} = \frac{1000}{0.0485343} = 20603.98$$

The amortization table is as follows:

Month	O/P	EMI	Interest Part	Principal Part
	(a)	(b)	(c) = (a) x i	(b) - (c)
1	100000	20603.98	1000	19603.98
2	80396.02	20603.98	803.96	19800.02
3	60596	20603.98	605.96	19998.02
4	40597.98	20603.98	405.98	20198
5	20399.98	20603.98	204	20399.98

Check your progress

1. An overdraft of Rs. 50,000 is to be paid back in equal annual installments in 20 years. Find the installments, if the interest is 12% p.a. compounded annually. $[(1.12)^{20} = 9.64629]$
2. A man borrows Rs. 30,000 at 6% p.a. compounded semi-annually for 5 years. Find the periodic payments he has to make.
3. What periodic payments Mr. Narayanan has to make if he has borrowed Rs. 1,00,000 at 12% p.a. compounded annually for 12 years? $[(1.12)^{12} = 3.896]$
4. Find the future value of an immediate annuity of Rs. 1200 at 6% p.a. compounded annually for 3 years.
5. Find the future value of an immediate annuity of Rs. 500 at 8% p.a. compounded p.m. for 5 years.
6. Find the accumulated value after 2 years if a sum of Rs. 1500 is invested at the end of every year at 10% p.a. compounded quarterly.
7. Find the accumulated amount of an immediate annuity of Rs. 1000 at 9% p.a. compounded semi-annually for 4 years.
8. Find the future value of an immediate annuity of Rs. 2800 paid at 10% p.a. compounded quarterly for 2 years. Also find the interest earned on the annuity.
9. Find the sum invested and the accumulated amount for an ordinary annuity with periodic payment of Rs. 2500, at the rate of interest of 9% p.a. for 2 years if the period of payment is (a) yearly, (b) half-yearly, (c) quarterly or (d) monthly.
10. Find the sum invested and the accumulated amount for an ordinary annuity with periodic payment of Rs. 1500, at the rate of interest of 10% p.a. for 3 years if the period of payment is (a) yearly, (b) half-yearly, (c) quarterly or (d) monthly.
11. Mr. Banerjee wants to accumulate Rs. 5,00,000 at the end of 10 years from now. How much amount should he invest every year at the rate of interest of 9% p.a. compounded annually?

12. Find the periodic payment to be made so that Rs. 25000 gets accumulated at the end of 4 years at 6% p.a. compounded annually.
13. Find the periodic payment to be made so that Rs. 30,000 gets accumulated at the end of 5 years at 8% p.a. compounded half yearly.
14. Find the periodic payment to be made so that Rs. 2000 gets accumulated at the end of 2 years at 12% p.a. compounded quarterly.
15. Find the rate of interest if a person depositing Rs. 1000 annually for 2 years receives Rs. 2070.
16. Find the rate of interest compounded p.a. if an immediate annuity of Rs. 50,000 amounts to Rs. 1,03,000 in 2 years.
17. Find the rate of interest compounded p.a. if an immediate annuity of Rs. 5000 amounts to Rs. 10400 in 2 years.
18. What is the value of the annuity at the end of 5 years, if Rs. 1000 p.m. is deposited into an account earning interest 9% p.a. compounded monthly? What is the interest paid in this amount?
19. What is the value of the annuity at the end of 3 years, if Rs. 500 p.m. is deposited into an account earning interest 6% p.a. compounded monthly? What is the interest paid in this amount?
20. Mr. Ashish Gokhale borrows Rs. 5000 from a bank at 8% compound interest. If he makes an annual payment of Rs. 1500 for 4 years, what is his remaining loan amount after 4 years?
(**Hint:** find the amount using compound interest formula for 4 years and then find the accumulated amount of annuity, the difference is the remaining amount.)
21. Find the present value of an immediate annuity of Rs. 10,000 for 3 years at 6% p.a. compounded annually.
22. Find the present value of an immediate annuity of Rs. 100000 for 4 years at 8% p.a. compounded half yearly.
23. Find the present value of an immediate annuity of Rs. 1600 for 2 years at 7% p.a. compounded half yearly.
24. A loan is repaid fully with interest in 5 annual installments of Rs. 15,000 at 8% p.a. Find the present value of the loan.
25. Mr. Suman borrows Rs. 50,000 from Mr. Juman and agreed to pay Rs. 14000 annually for 4 years at 10% p.a. Is this business profitable to Mr. Juman?
(**Hint:** Find the *PV* of the annuity and compare with Rs. 50000)
26. Mr. Paradkar is interested in saving a certain sum which will amount to Rs. 3,50,000 in 5 years. If the rate of interest is 12% p.a., how much should he save yearly to achieve his target?
27. Mr. Kedar Pethkar invests Rs. 10000 per year for his daughter from her first birthday onwards. If he receives an interest of 8.5% p.a., what is the amount accumulated when his daughter turns 18?

28. Dr. Wakankar, a dentist has started his own dispensary. He wants to install a machine chair which costs Rs. 3,25,000. The machine chair is also available on monthly rent of Rs. 9000 at 9% p.a. for 3 years. Should Dr. Wakankar buy it in cash or rent it?
29. A sum of Rs. 50,000 is required to buy a new machine in a factory. What sinking fund should the factory accumulate at 8% p.a. compounded annually if the machine is to be replaced after 5 years?
30. The present cost of a machine is Rs. 80,000. Find the sinking fund the company has to generate so that it could buy a new machine after 10 years, whose value then would be 25% more than of today's price. The rate of compound interest being 12% p.a. compounded annually.
31. Mr. Mistry has two options while buying a German wheel alignment machine for his garage: (a) either buy it at Rs. 1,26,000 or (b) take it on lease for 5 years at an annual rent of Rs. 30,000 at the rate of interest of 12% p.a.. Assuming no scrap value for the machine which option should Mr. Mistry exercise?
32. Regency Co-op. Hsg. Society which has 50 members require Rs. 12,60,000 at the end of 3 years from now for the society repairs. If the rate of compound interest is 10% p.a., how much fund the society should collect from every member to meet the necessary sum?
33. Mr. Lalwaney is of 40 years now and wants to create a fund of Rs. 15,00,000 when he is 60. What sum of money should he save annually so that at 13% p.a. he would achieve his target?
34. If a society accumulates Rs. 1000 p.a. from its 200 members for 5 years and receives 12% interest then find the sum accumulated at the end of the fifth year. If the society wants Rs. 13,00,000 for society maintenance after 5 years, is the annual fund of Rs. 1000 per member sufficient?
35. How much amount should a factory owner invest every year at 6% p.a. for 6 years, so that he can replace a mixture-drum (machine) costing Rs. 60,000, if the scrap value of the mixture-drum is Rs. 8,000 at the end of 6 years.
36. If a society accumulates Rs. 800 p.a. from its 100 members for 3 years and receives 9% interest then find the sum accumulated at the end of the third year. If the society wants Rs. 2,50,000 for society maintenance after 3 years, is the annual fund of Rs. 800 per member sufficient?
37. Mr. Kanishk wants clear his loan of Rs. 10,00,000 taken at 12% p.a. in 240 monthly installments. Find his EMI using reducing balance method.
38. Using the reducing balance method find the EMI for the following:

Loan amount (in Rs.)	Rate of Interest (in % p.a.)	Period of Loan (in years)
i) 1000	6	5
ii) 50000	6	10
iii) 8000	7	6
iv) 12000	9	10
v) 1000	9.5	10
vi) 1100000	12.5	20

39. Mr. Vilas Khopkar has taken a loan of Rs. 90,000 at 11% p.a. Find the EMI using (a) reducing balance method and (b) Flat interest method, if he has to return the loan in 4 years.
40. Find the EMI using reducing balance method on a sum of Rs. 36,000 at 9%, to be returned in 6 monthly installments.
41. Find the EMI using reducing balance method on a sum of Rs. 72,000 at 12%, to be returned in 12 installments.
42. Mr. Sachin Andhale has borrowed Rs. 10,000 from his friend at 9% p.a. and has agreed to return the amount with interest in 4 months. Find his EMI and also prepare the amortization table.
43. Mr. Arvind Kamble has borrowed Rs. 30,000 from his friend at 14% p.a. If he is to return this amount in 5 monthly installments, find the installment amount, the interest paid and prepare the amortization table for repayment.
44. Mrs. Chaphekar has taken a loan of Rs. 1,25,000 from a bank at 12% p.a. If the loan has to be returned in 3 years, find the EMI, Mrs. Chaphekar has to pay. Prepare the amortization table of repayment of loan and find the interest she has to pay.
45. A loan of Rs. 75,000 is to be returned with interest in 4 installments at 15% p.a. Find the value of the installments.
46. A loan of Rs. 60,000 is to be returned in 6 equal installments at 12% p.a. Find the amount of the installments.
47. Find the sum accumulated by paying an EMI of Rs. 11,800 for 2 years at 10% p.a.
48. Find the sum accumulated by paying an EMI of Rs. 1,800 for 2 years at 12% p.a.
49. Find the sum accumulated by paying an EMI of Rs. 12,000 for 3 years at 9% p.a.
50. Find the sum accumulated by paying an EMI of Rs. 11,000 for 8 years at 9.5% p.a.

Hints & Solutions to Check your progress

- (1) 6694 (2) 3517 (3) 16,144 (4) 3820.32
 (5) 36555.65 (6) 13104 (7) 9380 (8) 24461
 (9)

Period	Sum Invested	Accumulated Amount
Yearly	5000	5225
Half-yearly	10000	10695.5
Quarterly	20000	21648
Monthly	60000	65471

(10)

Period	Sum Invested	Accumulated Amount
Yearly	4500	4965
Half-yearly	9000	10203
Quarterly	18000	20693
Monthly	54000	62635

- (11) 32910 (12) 5715 (13) 2498.72 (14) 225
 (15) 7% (16) 6% (17) 8% (18) 75424, 15424
 (19) 19688, 1688 (20) 4719 (21) 26730
 (22) 673274.5 (23) 5877 (24) 59890.65 (25) 44378, Yes
 (26) 97093.4 (27) 393229.95 (28) 283021.25, take it on rent
 (29) 12523 (30) 17698.42 (31) 108143.28 < 126000, Mr. Mistry
 should use the second option. (32) 16245 (33) 18530
 (34) 1270569.47, not sufficient (35) 7454.86 (36) 2,62,248; yes
 (37) 11,011

(38)

Loan amount (in Rs.)	Rate of interest (in % p.a.)	Period of Loan (in yrs.)	EMI (in Rs.)
i) 1000	6	5	19
ii) 50000	6	10	555
iii) 8000	7	6	136
iv) 12000	9	10	152
v) 1000	9.5	10	13
vi) 1100000	12.5	20	12498

- (39) 2326, 2700 (40) 6158.48 (41) 6397.11

(42)

Month	O/P	EMI	Interest Part	Principal Part
	(a)	(b)	(c) = (a) x i	(b) - (c)
1	10000	2547.05	75	2472.05
2	7527.95	2547.05	56.45	2490.6
3	5037.35	2547.05	37.78	2509.27
4	2528.08	2547.05	18.96	2528.09

(43)

Month	O/P	EMI	Interest Part	Principal Part
	(a)	(b)	(c) = (a) x i	(b) - (c)
1	30000	6212.23	351	5861.23
2	24138.77	6212.23	282.42	5929.81
3	18208.96	6212.23	213.04	5999.19
4	12209.77	6212.23	142.85	6069.38
5	6140.39	6212.23	71.84	6140.39

(45) 19339.57

(46) 16353

(47) 3,12,673.60

(48) 48552.24

(49) 4,93,832.6

(50) 15,72,727



Unit-4

CORRELATION AND REGRESSION

Unit Structure :

- 4.0 Objectives
- 4.1 Introduction
- 4.2 Types of Correlation
- 4.3 Measurement of Correlation
- 4.4 Rank Correlation
- 4.5 Regression Analysis

4.0 OBJECTIVES

- To understand the relationship between two relevant characteristics of a statistical unit.
- Learn to obtain the numerical measure of the relationship between two variables.
- Use the mathematical relationship between two variables in order to estimate value of one variable from the other.
- Use the mathematical relationship to obtain the statistical constants line means and S.D.'s

4.1 INTRODUCTION

In the statistical analysis we come across the study of two or more relevant characteristics together in terms of their interrelations or interdependence. e.g. Interrelationship among production, sales and profits of a company. Inter relationship among rainfall, fertilizers, yield and profits to the farmers.

Relationship between price and demand of a commodity When we collect the information (data) on two of such characteristics it is called bivariate data. It is generally denoted by (X,Y) where X and Y are the variables representing the values on the characteristics.

Following are some examples of bivariate data.

- a) Income and Expenditure of workers.
- b) Marks of students in the two subjects of Maths and Accounts.
- c) Height of Husband and Wife in a couple.
- d) Sales and profits of a company.

Between these variables we can note that there exist some sort of interrelationship or cause and effect relationship. i.e. change in the value of one variable brings out the change in the value of other variable also. Such relationship is called as correlation.

Therefore, correlation analysis gives the idea about the nature and extent of relationship between two variables in the bivariate data.

4.2 TYPES OF CORRELATION:

There are two types of correlation.

- a) Positive correlation. and b) Negative correlation.

4.2.1 Positive correlation: When the relationship between the variables X and Y is such that increase or decrease in X brings out the increase or decrease in Y also, i.e. there is direct relation between X and Y , the correlation is said to be positive. In particular when the 'change in X equals to change in Y ' the correlation is perfect and positive. e.g. Sales and Profits have positive correlation.

4.2.2 Negative correlation: When the relationship between the variables X and Y is such that increase or decrease in X brings out the decrease or increase in Y , i.e. there is an inverse relation between X and Y , the correlation is said to be negative. In particular when the 'change in X equals to change in Y ' but in opposite direction the correlation is perfect and negative. e.g. Price and Demand have negative correlation.

4.3 MEASUREMENT OF CORRELATION

The extent of correlation can be measured by any of the following methods:

- Scatter diagrams
- Karl Pearson's co-efficient of correlation
- Spearman's Rank correlation

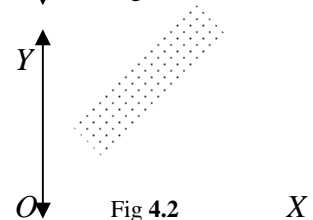
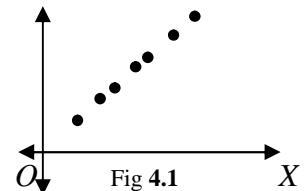
4.3.1 Scatter Diagram: The Scatter diagram is a chart prepared by plotting the values of X and Y as the points (X,Y) on the graph. The pattern of the points is used to explain the nature of correlation as follows. The following figures and the explanations would make it clearer.

(i) *Perfect Positive Correlation:*

If the graph of the values of the variables is a straight line with positive slope as shown in Figure 4.1, we say there is a *perfect positive correlation* between X and Y . Here $r = 1$.

(ii) *Imperfect Positive Correlation:*

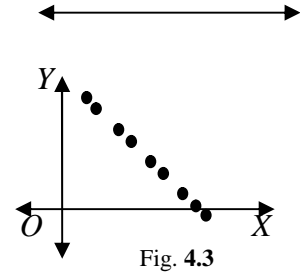
If the graph of the values of X and Y show a band of points from lower left corner to upper right corner as shown in Figure 4.2, we say that there is an *imperfect*



positive correlation. Here $0 < r < 1$.

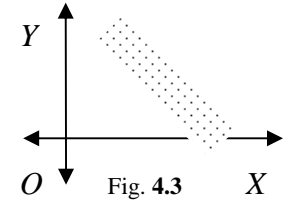
(iii) *Perfect Negative Correlation:*

If the graph of the values of the variables is a straight line with negative slope as shown in Figure 4.3, we say there is a *perfect negative correlation* between X and Y. Here $r = -1$.



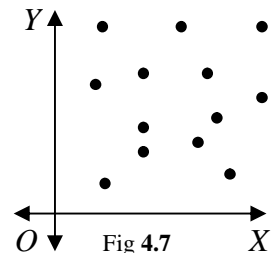
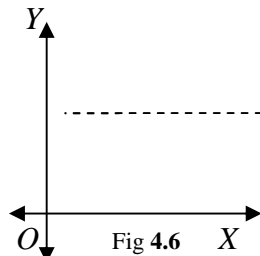
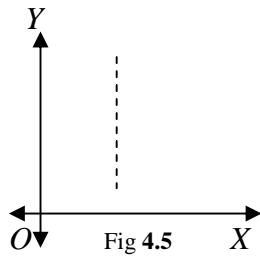
(iv) *Imperfect Negative Correlation:*

If the graph of the values of X and Y show a band of points from upper left corner to the lower right corner as shown in Figure 4.4, then we say that there is an *imperfect negative correlation*. Here $-1 < r < 0$



(v) *Zero Correlation:*

If the graph of the values of X and Y do not show any of the above trend then we say that there is a *zero correlation* between X and Y. The graph of such type can be a straight line perpendicular to the axis, as shown in Figure 4.5 and 4.6, or may be completely scattered as shown in Figure 4.7. Here $r = 0$.



The Figure 4.5 show that the increase in the values of Y has no effect on the value of X, it remains the same, hence zero correlation. The Figure 4.6 show that the increase in the values of X has no effect on the value of Y, it remains the same, hence zero correlation. The Figure 4.7 show that the points are completely scattered on the graph and show no particular trend, hence there is no correlation or zero correlation between X and Y.

4.3.2 Karl Pearson’s co-efficient of correlation.

This co-efficient provides the numerical measure of the correlation between the variables X and Y. It is suggested by Prof. Karl Pearson and calculated by the formula

$$r = \frac{Cov(x, y)}{\sigma_x \cdot \sigma_y}$$

Where, $Cov(x,y)$: Covariance between x & y

σ_x : Standard deviation of x & σ_y : Standard deviation of y

Also, $Cov(x,y) = \frac{1}{n} \sum (x - \bar{x}) (y - \bar{y}) = \frac{1}{n} \sum xy - \bar{x} \bar{y}$

$$\text{S.D.}(x) = \sigma_x = \sqrt{\frac{1}{n} \Sigma(x-\bar{X})^2} = \sqrt{\frac{1}{n} \Sigma x^2 - \bar{X}^2} \quad \text{and}$$

$$\text{S.D.}(y) = \sigma_y = \sqrt{\frac{1}{n} \Sigma(y-\bar{Y})^2} = \sqrt{\frac{1}{n} \Sigma y^2 - \bar{Y}^2}$$

Remark : We can also calculate this co-efficient by using the formula given by

$$r = \frac{\frac{1}{n} \Sigma(x-\bar{X})(y-\bar{Y})}{\sqrt{\frac{1}{n} \Sigma(x-\bar{X})^2} \sqrt{\frac{1}{n} \Sigma(y-\bar{Y})^2}} = \frac{\frac{\Sigma xy}{n} - \bar{X}\bar{Y}}{\sqrt{\left(\frac{\Sigma x^2}{n} - \bar{X}^2\right) \left(\frac{\Sigma y^2}{n} - \bar{Y}^2\right)}}$$

The Pearson's Correlation co-efficient is also called as the 'product moment correlation co-efficient'

Properties of correlation co-efficient 'r'

The value of 'r' can be positive (+) or negative(-)

The value of 'r' always lies between -1 & +1, i.e. $-1 < r < +1$

Significance of 'r' equals to -1, +1 & 0

When 'r' = +1; the correlation is perfect and positive.

'r' = -1; the correlation is perfect and negative.

and when there is no correlation 'r' = 0

SOLVED EXAMPLES :

Example.1: Calculate the Karl Pearson's correlation coefficient from the following.

X:	12	10	20	13	15
Y:	7	14	6	12	11

Solution: Table of calculation,

X	Y	XxY	X ²	Y ²	
12	7	84	144	49	
10	14	140	100	196	
20	6	120	400	36	
13	12	156	169	144	
15	11	165	225	121	
$\underline{\Sigma x} = 70$	$\underline{\Sigma y} = 50$	$\underline{\Sigma xy} = 665$	$\underline{\Sigma x^2} = 1038$	$\underline{\Sigma y^2} = 546$	And n = 5

The Pearson's correlation coefficient r is given by,

$$r = \frac{\text{Cov}(x, y)}{\sigma_x \cdot \sigma_y}$$

Where,

$$\bar{x} = \frac{\Sigma x}{n} = \frac{70}{5} = 14 \quad \bar{y} = \frac{\Sigma y}{n} = \frac{50}{5} = 10$$

$$\begin{aligned} \text{Cov}(x,y) &= \frac{\Sigma xy}{n} - \bar{X}\bar{Y} & \sigma_x &= \sqrt{\frac{1}{n}\Sigma x^2 - \bar{X}^2} & \sigma_y &= \sqrt{\frac{1}{n}\Sigma y^2 - \bar{Y}^2} \\ &= \frac{665}{5} - 14 \times 10 & &= \sqrt{\frac{1038}{5} - 14^2} & &= \sqrt{\frac{546}{5} - 10^2} \\ &= 133 - 140 & &= \sqrt{11.6} = 3.40 & &= \sqrt{9.2} = 3.03 = -0.7 \end{aligned}$$

$$\therefore \text{Cov}(x,y) = -7 \quad \sigma_x = 3.40 \quad \text{and} \quad \sigma_y = 3.03$$

Substituting the values in the formula of r we get

$$r = \frac{-7}{3.40 \times 3.03} = -0.68$$

$$\therefore r = -0.68$$

Example 2: Let us calculate co-efficient of correlation between Marks of students in the Subjects of Maths & Accounts. in a certain test conducted.

Table of calculation:

Marks In Maths X	Marks In Accounts Y	XY	X ²	Y ²
28	30	840	784	900
25	40	1000	625	1600
32	50	1600	1024	2500
16	18	288	256	324
20	25	500	400	625
15	12	180	225	144
19	11	209	361	121
17	21	357	289	441
40	45	1800	1600	2025
30	35	1050	900	1225
$\Sigma x = 242$	$\Sigma y = 287$	$\Sigma xy = 7824$	$\Sigma x^2 = 6464$	$\Sigma y^2 = 9905$

$$n=10$$

Now Pearson's co-efficient of correlation is given by the formula,

$$r = \frac{\text{Cov}(x,y)}{\sigma_x \cdot \sigma_y}$$

Where,

$$\bar{x} = \frac{\Sigma x}{n} = \frac{242}{10} = 24.2 \quad \bar{y} = \frac{\Sigma y}{n} = \frac{287}{10} = 28.7$$

$$\begin{array}{l}
\text{Cov}(x,y) = \frac{\Sigma xy}{n} - \bar{X}\bar{Y} \\
= \frac{7824}{10} - 24.2 \times 28.7 \\
= 782.4 - 694.54
\end{array}
\left|
\begin{array}{l}
\sigma_x = \sqrt{\frac{1}{n} \Sigma x^2 - \bar{X}^2} \\
\sigma_x = \sqrt{\frac{6464}{10} - 24.2^2} \\
= \sqrt{60.76}
\end{array}
\right|
\begin{array}{l}
\sigma_y = \sqrt{\frac{1}{n} \Sigma y^2 - \bar{Y}^2} \\
\sigma_y = \sqrt{\frac{9905}{10} - 28.7^2} \\
= \sqrt{166.81}
\end{array}$$

$$\text{Cov}(x,y) = 87.86, \quad \sigma_x = 7.79 \text{ and } \sigma_y = 12.91$$

$\therefore \text{Cov}(x,y) = 87.86 \quad \sigma_x = 7.79 \quad \text{and} \quad \sigma_y = 12.91$
Substituting the values in the formula of r we get

$$r = \frac{87.86}{7.79 \times 12.91} = 0.87$$

$$\therefore r = 0.87$$

4.4 RANK CORRELATION

In many practical situations, we do not have the scores on the characteristics, but the ranks (preference order) decided by two or more observers. Suppose, a singing competition of 10 participants is judged by two judges A and B who rank or assign scores to the participants on the basis of their performance. Then it is quite possible that the ranks or scores assigned may not be equal for all the participants. Now the difference in the ranks or scores assigned indicates that there is a difference of opinion between the judges on deciding the ranks. The rank correlation studies the association in this ranking of the observations by two or more observers. The measure of the extent of association in rank allocation by the two judges is calculated by the co-efficient of Rank correlation 'R'. This co-efficient was developed by the British psychologist Edward Spearman in 1904.

Mathematically, Spearman's rank correlation co-efficient is defined as,

$$R = 1 - \frac{6 \Sigma d^2}{n(n^2 - 1)}$$

Where d = rank difference and n = no of pairs.

Remarks: We can note that, the value of 'R' always lies between -1 and $+1$

The positive value of 'R' indicates the positive correlation (association) in the rank allocation. Whereas, the negative value of 'R' indicates the negative correlation (association) in the rank allocation.

SOLVED EXAMPLES:**Example 3****a) When ranks are given:-**

Data given below read the ranks assigned by two judges to 8 participants. Calculate the co-efficient of Rank correlation.

Participant No.	Ranks by Judge		Rank diff Square d^2
	A	B	
1	5	4	$(5-4)^2 = 1$
2	6	8	4
3	7	1	36
4	1	7	36
5	8	5	9
6	2	6	16
7	3	2	1
8	4	3	1
N = 8	Total		$104 = \Sigma d^2$

Spearman's rank correlation co-efficient is given by

$$R = 1 - \frac{6\Sigma d^2}{n(n^2 - 1)}$$

Substituting the values from the table we get,

$$R = 1 - \frac{6 \times 104}{8(8^2 - 1)} = -0.23$$

The value of correlation co-efficient is - 0.23. This indicates that there is negative association in rank allocation by the two judges A and B

b) When scores are given:-**Example 4**

The data given below are the marks given by two Examiners to a set of 10 students in a aptitude test. Calculate the Spearman's Rank correlation co-efficient, 'R'

Now the Spearman's rank correlation co-efficient is given by

$$R = 1 - \frac{6\Sigma d^2}{n(n^2 - 1)}$$

Substituting the values from the table we get,

$$\begin{aligned} R &= 1 - \frac{6 \times 5}{10(10^2 - 1)} \\ &= 1 - 0.03 \\ &= 0.97 \end{aligned}$$

The value of correlation co-efficient is + 0.97. This indicates that there is positive association in assessment of two examiners, A and B.

c) Case of repeated values:-

It is quite possible that the two participants may be assigned the same score by the judges. In such cases Rank allocation and calculation of rank correlation can be explained as follows.

Example: The data given below scores assigned by two judges for 10 participants in the singing competition. Calculate the Spearman's Rank correlation co-efficient.

Participant No.	Score assigned By Judges		Ranks		Rank difference square
	A	B	R _A	R _B	D ²
1	28	35	<u>9 (8.5)</u>	6	(8.5-6) ² =6.25
2	40	26	3	<u>10(9.5)</u>	42.25
3	35	42	<u>5 (4.5)</u>	3	2.25
4	25	26	10	<u>9 (9.5)</u>	0.25
5	28	33	<u>8 (8.5)</u>	7	2.25
6	35	45	<u>4 (4.5)</u>	2	6.25
7	50	32	1	8	49
8	48	51	2	1	1
9	32	39	6	4	4
10	30	36	7	5	4
N = 10				Total	Σd ² =117.5

Student No.	Marka By Examiner		Ranks		Rank difference square
	A	B	R _A	R _B	D ²
1	85	80	2	2	0
2	56	60	8	7	1
3	45	50	10	10	0
4	65	62	6	6	0
5	96	90	1	1	0
6	52	55	9	8	1
7	80	75	3	4	1
8	75	68	5	5	0
9	78	77	4	3	1
10	60	53	7	9	1
N = 10				Total	5 = Σd ²

Explanation:- In the column of A and B there is repetition of scores so while assigning the ranks we first assign the ranks by treating them as different values and then for repeated scores we assign the average rank. e.g In col A the score 35 appears 2 times at number 4 and 5 in the order of ranking so we calculate the average rank as $(4+5)/2 = 4.5$. Hence the ranks assigned are 4.5 each. The other repeated scores can be ranked in the same manner.

Note: In this example we can note that the ranks are in fraction e.g. 4.5, which is logically incorrect or meaningless. Therefore in the calculation of 'R' we add a correction factor (C.F.) to Σd^2 calculated as follows.

Table of correction factor (C.F.)

Value Repeated	Frequency M	$m(m^2-1)$
35	2	$2 \times (2^2-1)=6$
28	2	6
26	2	6
	Total	$\Sigma m(m^2-1)=18$

$$\text{Now } C.F. = \frac{\Sigma(m^3 - m)}{12} = 18/12 = 1.5$$

$$\therefore \Sigma d^2 = 117.5 + 1.5 = 119$$

We use this value in the calculation of 'R'

Now the Spearman's rank correlation co-efficient is given by

$$R = 1 - \frac{6\Sigma d^2}{n(n^2 - 1)}$$

$$\text{Substituting the values we get, } R = 1 - \frac{6 \times 119}{10(10^2 - 1)} = 1 - 0.72 = 0.28$$

EXERCISE I

1. What is mean by correlation? Explain the types of correlation with suitable examples.
2. What is a scatter diagram? Draw different scattered diagrams to explain the correlation between two variables x and y.
3. State the significance of 'r' = +1, -1 and 0.
4. Calculate the coefficient of correlation r from the following data.
 X: 18 12 16 14 10 15 17 13
 Y: 9 13 20 15 11 24 26 22

5. The following table gives the price and demand of a certain commodity over the period of 8 months. Calculate the Pearson's coefficient of correlation.

Price:	15	12	23	25	18	17	11	19
Demand	45	30	60	65	48	45	28	50

6. Following results are obtained on a certain bivariate data.

(i) $n = 10$ $\Sigma x = 75$ $\Sigma y = 70$ $\Sigma x^2 = 480$

$\Sigma y^2 = 600$ $\Sigma xy = 540$

(ii) $n = 15$ $\Sigma x = 60$ $\Sigma y = 85$ $\Sigma x^2 = 520$

$\Sigma y^2 = 1200$ $\Sigma xy = -340$

Calculate the Pearson's correlation coefficient in each case.

7. Following data are available on a certain bi-variate data :

(i) $\Sigma(x - \bar{x})(y - \bar{y}) = 120$, $\Sigma(x - \bar{x})^2 = 150$ $\Sigma(y - \bar{y})^2 = 145$

(ii) $\Sigma(x - \bar{x})(y - \bar{y}) = -122$, $\Sigma(x - \bar{x})^2 = 136$ $\Sigma(y - \bar{y})^2 = 148$

Find the correlation coefficient.

8. Calculate the Pearson's coefficient of correlation from the given information on a bivariate series:

No of pairs: 25

Sum of x values: 300

Sum of y values: 375

Sum of squares of x values: 9000

Sum of squares of y values: 6500

Sum of the product of x and y values: 4000.

9. The ranks assigned to 8 participants by two judges are as follows. Calculate the Spearman's Rank correlation coefficient 'R'.

Participant No: 1 2 3 4 5 6 7 8

Ranks by Judge I: 5 3 4 6 1 8 7 2

Judge II: 6 8 3 7 1 5 4 2

10. Calculate the coefficient of rank correlation from the data given below.

X: 40 33 60 59 50 55 48

Y: 70 60 85 75 72 82 69

11. Marks given by two Judges to a group of 10 participants are as follows. Calculate the coefficient of rank correlation.

Marks by Judge

A: 52 53 42 60 45 41 37 38 25 27

Judge B: 65 68 43 38 77 48 35 30 25 50.

12. An examination of 8 applicants for a clerical post was by a bank. The marks obtained by the applicants in the subjects of Mathematics and Accountancy were as follows. Calculate the rank correlation coefficient.

Applicant: A B C D E F G H

Marks in								
Maths:	15	20	28	12	40	60	20	80
Marks in								
Accounts:	40	30	50	30	20	10	25	60

4.5 REGRESSION ANALYSIS

As the correlation analysis studies the nature and extent of interrelationship between the two variables X and Y, regression analysis helps us to estimate or approximate the value of one variable when we know the value of other variable. Therefore we can define the 'Regression' as the estimation (prediction) of one variable from the other variable when they are correlated to each other. e.g. We can estimate the Demand of the commodity if we know it's Price.

Why are there two regressions?

When the variables X and Y are correlated there are two possibilities,

- Variable X depends on variable y. in this case we can find the value of x if know the value of y. This is called regression of x on γ .
- Variable γ depends on variable X. we can find the value of y if know the value of X. This is called regression of y on x. Hence there are two regressions,

(a) Regression of X on Y; (b) Regression of X on Y.

4.5.1 Formulas on Regression equation,

Regression of X on Y	Regression of X on Y
Assumption: X depends on Y The regression equation is $(x - \bar{x}) = b_{xy}(y - \bar{y})$	Y depends on X The regression equation is $(y - \bar{y}) = b_{yx}(x - \bar{x})$
b_{xy} = Regression co-efficient of X on Y = $\frac{Cov(x, y)}{V(y)}$	b_{yx} = Regression co-efficient of Y on X = $\frac{Cov(x, y)}{V(x)}$

Where,

$$Cov(x, y) = \frac{1}{n} \sum (x - \bar{x})(y - \bar{y}) = \frac{1}{n} \sum xy - \bar{x} \bar{y}$$

$$V(x) = \frac{1}{n} \sum (x - \bar{x})^2 \quad \text{and} \quad V(y) = \frac{1}{n} \sum (y - \bar{y})^2$$

$$V(x) = \frac{1}{n} \sum x^2 - \bar{x}^2 \quad \text{and} \quad V(y) = \frac{1}{n} \sum y^2 - \bar{y}^2$$

Use: To find X

Use: To find γ

SOLVED EXAMPLES**Example 1:**

Obtain the two regression equations and hence find the value of x when y=25

Data:-

X	Y	X ²	Y ²	XxY
8	15	64	225	120
10	20	100	400	200
12	30	144	900	360
15	40	225	1600	600
20	45	400	2025	900
$\underline{\Sigma x = 65}$	$\underline{\Sigma y = 150}$	$\underline{\Sigma x^2 = 933}$	$\underline{\Sigma y^2 = 5150}$	$\underline{\Sigma xy = 2180}$

And n= 5

Now the two regression equations are,

$$(x - \bar{x}) = b_{xy}(y - \bar{y}) \quad \text{-----x on y (i)}$$

$$(y - \bar{y}) = b_{yx}(x - \bar{x}) \quad \text{-----y on x (ii)}$$

Where,

$$\bar{x} = \frac{1}{n} \Sigma x = \frac{65}{5} = 13 \quad \text{and} \quad \bar{y} = \frac{1}{n} \Sigma y = \frac{150}{5} = 30$$

Also,

$$\begin{array}{l} \text{Cov}(x,y) = \frac{1}{n} \Sigma xy - \bar{x} \bar{y} \\ = \frac{2180}{5} - 13 \times 30 \\ = 436 - 390 \\ \therefore \text{Cov}(x,y) = 46 \end{array} \quad \left| \begin{array}{l} V(x) = \frac{1}{n} \Sigma x^2 - \bar{x}^2 \\ = \frac{933}{5} - 13^2 \\ = 186.6 - 169 \\ V(x) = 17.6 \end{array} \right| \quad \left| \begin{array}{l} V(y) = \frac{1}{n} \Sigma y^2 - \bar{y}^2 \\ = \frac{5150}{5} - 30^2 \\ = 1030 - 900 \\ V(y) = 130 \end{array} \right.$$

Now we find,

Regression co-efficient of X on Y | Regression co-efficient of X on Y

$$\begin{array}{l} b_{xy} = \frac{\text{Cov}(x,y)}{V(y)} \\ = \frac{46}{130} \\ \therefore b_{xy} = 0.35 \text{ and} \end{array} \quad \left| \begin{array}{l} b_{yx} = \frac{\text{Cov}(x,y)}{V(x)} \\ = \frac{46}{17.6} \\ b_{yx} = 2.61 \end{array} \right.$$

Now substituting the values of \bar{x} , \bar{y} , b_{xy} and b_{yx} in the regression equations we get,

$$(x-13) = 0.35(y-30) \quad \text{-----x on y (i)}$$

$$(y-30) = 2.61(x-13) \quad \text{-----y on x (ii)}$$

as the two regression equations.

Now to estimate x when y = 25, we use the regression equation of x on y

$$\therefore (x-13) = 0.35(25-30)$$

$$\therefore x = 13 - 1.75 = 11.25$$

Remark:

From the above example we can note some points about Regression coefficients.

- Both the regression coefficients carry the same sign (+ or -)
- Both the regression coefficients can not be greater than 1 in number
(e.g. -1.25 and -1.32) is not possible.
- Product of both the regression coefficients b_{xy} and b_{yx} must be < 1
i.e. $b_{xy} \times b_{yx} < 1$ Here $0.35 \times 2.61 = 0.91 < 1$ (**Check this always**)

Example 2:

Obtain the two regression equations and hence find the value of y when x=10

Data:-

X	Y	XxY	X ²	Y ²
12	25	300	144	625
20	18	360	400	324
8	17	136	64	289
14	13	182	196	169
16	15	240	256	225
$\Sigma x = 70$	$\Sigma y = 88$	$\Sigma xy = 1218$	$\Sigma x^2 = 1060$	$\Sigma y^2 = 1632$

And n= 5

Now the two regression equations are,

$$(x - \bar{x}) = b_{xy}(y - \bar{y}) \quad \text{-----x on y (i)}$$

$$(y - \bar{y}) = b_{yx}(x - \bar{x}) \quad \text{-----y on x (ii)}$$

Where,

$$\bar{x} = \frac{1}{n} \Sigma x = \frac{70}{5} = 14 \quad \text{and} \quad \bar{y} = \frac{1}{n} \Sigma y = \frac{88}{5} = 17.6$$

Also,

$$\begin{array}{l} \text{Cov}(x,y) = \frac{1}{n} \Sigma xy - \bar{x} \bar{y} \\ = \frac{1218}{5} - 14 \times 17.6 \\ = 243.6 - 246.4 \\ \therefore \text{Cov}(x,y) = -2.8 \end{array} \quad \left| \begin{array}{l} V(x) = \frac{1}{n} \Sigma x^2 - \bar{x}^2 \\ = \frac{1060}{5} - 14^2 \\ = 212 - 196 \\ V(x) = 16 \end{array} \right| \quad \left| \begin{array}{l} V(y) = \frac{1}{n} \Sigma y^2 - \bar{y}^2 \\ = \frac{1632}{5} - 17.6^2 \\ = 326.4 - 309.76 \\ V(y) = 16.64 \end{array} \right.$$

Now we find,

<p>Regression co-efficient of X on Y</p> $b_{xy} = \frac{Cov(x, y)}{V(y)}$ $= \frac{2.8}{16.64}$ $\therefore b_{xy} = -0.168$		<p>Regression co-efficient of X on Y</p> $b_{yx} = \frac{Cov(x, y)}{V(x)}$ $= -\frac{2.8}{16.64}$ $b_{yx} = 0.175$
---	--	--

Now substituting the values of \bar{x} , \bar{y} , b_{xy} and b_{yx} in the regression equations we get,

$$(x-14) = -0.168(y-17.6) \text{ -----x on y (i)}$$

$$(y-17.6) = -0.175(x-14) \text{ ----- y on x (ii)}$$

as the two regression equations.

Now to estimate y when x =10, we use the regression equation of y on x

$$\therefore (y-17.6) = -0.175(10-14)$$

$$\therefore y = 17.6 + 0.7 = 24.3$$

Example 3:

The following data give the experience of machine operators and their performance rating given by the number of good parts turned out per 100 pieces.

Operator:	1	2	3	4	5	6	7	8
Experience:	16	12	18	4	3	10	5	12
(in years)								
Performance:	87	88	89	68	78	80	75	83
Rating								

Obtain the two regression equations and estimate the performance rating of an operator who has put 15 years in service.

Solution: We define the variables,

X: Experience y: Performance rating

Table of calculations:

X	Y	Xy	x ²	Y ²
16	87	1392	256	7569
12	88	1056	144	7744
18	89	1602	324	7921
4	68	272	16	4624
3	78	234	9	6084
10	80	800	100	6400
5	75	375	25	5625
12	83	996	144	6889
<u>Σx= 80</u>	<u>Σy= 648</u>	<u>Σxy= 6727</u>	<u>Σx²= 1018</u>	<u>Σy²= 52856</u>

Now the two regression equations are,

$$(x - \bar{x}) = b_{xy}(y - \bar{y}) \quad \text{-----x on y (i)}$$

$$(y - \bar{y}) = b_{yx}(x - \bar{x}) \quad \text{----- y on x (ii)}$$

Where,

$$\bar{x} = \frac{1}{n} \sum x = \frac{80}{8} = 10 \quad \text{and} \quad \bar{y} = \frac{1}{n} \sum y = \frac{648}{8} = 81$$

Also,

$$\begin{array}{l} \text{Cov}(x,y) = \frac{1}{n} \sum xy - \bar{x} \bar{y} \\ \quad = \frac{6727}{8} - 10 \times 81 \\ \quad = 840.75 - 810 \\ \therefore \text{Cov}(x,y) = 30.75 \end{array} \quad \left| \begin{array}{l} V(x) = \frac{1}{n} \sum x^2 - \bar{x}^2 \\ \quad = \frac{1018}{8} - 10^2 \\ \quad = 127.25 - 100 \\ V(x) = 27.25 \end{array} \right| \quad \left| \begin{array}{l} V(y) = \frac{1}{n} \sum y^2 - \bar{y}^2 \\ \quad = \frac{52856}{8} - 81^2 \\ \quad = 6607 - 6561 \\ V(y) = 46 \end{array} \right.$$

Now we find,

Regression co-efficient of X on Y Regression co-efficient of X on Y

$$\begin{array}{l} b_{xy} = \frac{\text{Cov}(x,y)}{V(y)} \\ \quad = \frac{30.75}{46} \\ \therefore b_{xy} = 0.67 \end{array} \quad \left| \quad \begin{array}{l} b_{yx} = \frac{\text{Cov}(x,y)}{V(x)} \\ \quad = \frac{30.75}{27.25} \\ \therefore b_{yx} = 1.13 \end{array} \right.$$

Now substituting the values of \bar{x} , \bar{y} , b_{xy} and b_{yx} in the regression equations we get,

$$(x-10) = 0.67(y-81) \quad \text{-----x on y (i)}$$

$$(y-81) = 1.13(x-10) \quad \text{----- y on x (ii)}$$

as the two regression equations.

Now to estimate Performance rating (y) when Experience (x) = 15, we use the regression equation of y on x

$$\therefore (y-81) = 1.13(15-10)$$

$$\therefore y = 81 + 5.65 = 86.65$$

Hence the estimated performance rating for the operator with 15 years of experience is approximately 86.65 i.e approximately 87

4.5.2 Regression coefficients in terms of correlation coefficient.

We can also obtain the regression coefficients b_{xy} and b_{yx} from standard deviations, σ_x , σ_y and correlation coefficient 'r' using the formulas

$$b_{xy} = r \frac{\sigma_x}{\sigma_y} \quad \text{and} \quad b_{yx} = r \frac{\sigma_y}{\sigma_x}$$

Also consider,

$$b_{xy} \times b_{yx} = r \frac{\sigma_x}{\sigma_y} r \frac{\sigma_y}{\sigma_x} = r^2 \quad \text{i.e. } r = \sqrt{b_{xy} \times b_{yx}}$$

Hence the correlation coefficient 'r' is the geometric mean of the regression coefficients, b_{xy} and b_{yx}

Example 5:

You are given the information about advertising expenditure and sales:

Exp. on Advertisement (Rs. In Lakh)	Sales (Rs. In Lakh)	
Mean	10	90
S.D.	3	12

Coefficient of correlation between sales and expenditure on Advertisement is 0.8. Obtain the two regression equations.

Find the likely sales when advertisement budget is Rs. 15 Lakh.

Solution: We define the variables,

X: Expenditure on advertisement

Y: Sales achieved.

Therefore we have,

$$\bar{x} = 10, \bar{y} = 90, \sigma_x = 3, \sigma_y = 12 \text{ and } r = 0.8$$

Now, using the above results we can write the two regression equations as

$$(x - \bar{x}) = r \frac{\sigma_x}{\sigma_y} (y - \bar{y}) \quad \text{-----x on y (i)}$$

$$(y - \bar{y}) = r \frac{\sigma_y}{\sigma_x} (x - \bar{x}) \quad \text{-----y on x (ii)}$$

Substituting the values in the equations we get,

$$(x - 10) = 0.8 \frac{3}{12} (y - 90)$$

$$\text{i.e. } x - 10 = 0.2 (y - 90) \quad \text{-----x on y (i)}$$

$$\text{also } (y - 90) = 0.8 \frac{12}{3} (x - 10)$$

$$\text{i.e. } y - 90 = 3.2 (x - 10) \quad \text{-----y on x (ii)}$$

Now when expenditure on advertisement (x) is 15, we can find the sales from eqn (ii) as,

$$y - 90 = 3.2 (15 - 10)$$

$$\therefore y = 90 + 16 = 106$$

Thus the likely sales are Rs.106 Lakh.

Example 6: Compute the two regression equations on the basis of the following information:

	X	Y
Mean	40	45
Standard deviation	10	9

Karl Pearson's coefficient of correlation between x and y = 0.50.
Also estimate the value of x when y = 48 using the appropriate equation.

Solution: We have,

$$\bar{x} = 40, \bar{y} = 45, \sigma_x = 10, \sigma_y = 9 \text{ and } r = 0.5$$

Now, we can write the two regression equations as

$$(x - \bar{x}) = r \frac{\sigma_x}{\sigma_y} (y - \bar{y}) \text{ -----x on y (i)}$$

$$(y - \bar{y}) = r \frac{\sigma_y}{\sigma_x} (x - \bar{x}) \text{ ----- y on x (ii)}$$

Substituting the values in the equations we get,

$$(x - 40) = 0.5 \frac{10}{9} (y - 45)$$

i.e. $x - 40 = 0.55 (y - 45)$ -----eqn of x on y (i)

and $(y - 45) = 0.5 \frac{9}{10} (x - 40)$

i.e. $y - 45 = 0.45(x - 40)$ -----eqn of y on x (ii)

Now when y is 48, we can find x from eqn (i) as,

$$x - 40 = 0.55(48 - 45)$$

$$\therefore x = 40 + 1.65 = 41.65$$

Example 7:

Find the marks of a student in the Subject of Mathematics who have scored 65 marks in Accountancy Given,

Average marks in Mathematics	70
Accountancy	80
Standard Deviation of marks in Mathematics	8
in Accountancy	10

Coefficient of correlation between the marks of Mathematics and marks of Accountancy is 0.64.

Solution: We define the variables,
 X: Marks in Mathematics
 Y: Marks in Accountancy

Therefore we have,

$$\bar{x} = 70, \bar{y} = 80, \sigma_x = 8, \sigma_y = 10 \text{ and } r = 0.64$$

Now we want to approximate the marks in Mathematics (x), we obtain the regression equation of x on y, which is given by

$$(x - \bar{x}) = r \frac{\sigma_x}{\sigma_y} (y - \bar{y}) \text{ -----x on y (i)}$$

Substituting the values we get,

$$(x - 70) = 0.64 \frac{8}{10} (y - 80)$$

i.e $x - 70 = 0.57 (y - 80)$

Therefore, when marks in Accountancy (Y) = 65

$$x - 70 = 0.57(65 - 80)$$

$$\therefore x = 70 - 2.85 = 67.15 \quad \text{i.e. } 67 \text{ approx.}$$

Use of regression equations to find means \bar{x}, \bar{y} S.D.s σ_x, σ_y and correlation coefficient 'r'

As we have that, we can obtain the regression equations from the values of Means, standard deviations and correlation coefficients 'r', we can get back these values from the regression equations.

Now, we can note that the regression equation is a linear equation in two variables x and y. Therefore, the linear equation of the type $Ax + By + C = 0$ or $y = a + bx$ represents a regression equation.

e.g. $3x + 5y - 15 = 0$ and $2x + 7y + 10 = 0$ represent the two regression equations.

The values of means \bar{x}, \bar{y} can be obtain by solving the two equations as the simultaneous equations.

Example 8:

From the following regression equation, find **means** \bar{x}, \bar{y} , σ_x, σ_y and 'r'

$$3x - 2y - 10 = 0, 24x - 25y + 145 = 0$$

Solution: The two regression equations are,

$$3x - 2y - 10 = 0 \text{ -----(i)}$$

$$24x - 25y + 145 = 0 \text{ ---(ii)}$$

Now for \bar{x} and \bar{y} we solve the two equations as the simultaneous equations.

Therefore, by (i) x 8 and (ii) x 1, we get

$$\begin{aligned} 24x - 16y - 80 &= 0 \\ 24x - 25y + 145 &= 0 \end{aligned}$$

$$\begin{array}{r} - \quad + \quad - \\ \hline 9y - 225 = 0 \end{array} \qquad y = \frac{225}{9} = 25$$

Putting $y = 25$ in eqn (i), we get

$$\begin{aligned} 3x - 2(25) - 10 &= 0 \\ 3x - 60 &= 0 \end{aligned} \qquad x = \frac{60}{3} = 20$$

Hence $\bar{x} = 20$ and $\bar{y} = 25$.

Now to find 'r' we express the equations in the form $y = a + bx$

So, from eqns (i) and (ii)

$$y = \frac{3x}{2} - \frac{10}{2} \qquad \text{and} \qquad y = \frac{24x}{25} + \frac{145}{25}$$

$$\therefore b_1 = \frac{3}{2} = 1.5 \qquad \therefore b_2 = \frac{24}{25} = 0.96$$

Since, $b_1 > b_2$ (i.e. b_2 is smaller in number irrespective of sign + or -)

\therefore Equation (ii) is regression of y on x and $b_{yx} = 0.96$

Hence eqn (i) is regression of x on y and $b_{xy} = 1/1.5 = 0.67$

$$\text{Now we find, } r = \sqrt{b_{xy} \times b_{yx}} \qquad \text{i.e. } r = \sqrt{0.67 \times 0.96} = +0.84$$

(The sign of 'r' is same as the sign of regression coefficients)

Example 9:

Find the means values of x, y , and r from the two regression equations.
 $3x + 2y - 26 = 0$ and $\sigma_x + y - 31 = 0$. Also find σ_x when $\sigma_y = 3$.

Solution: The two regression equations are,

$$\begin{aligned} 3x + 2y - 26 &= 0 \text{ ----- (i)} \\ 6x + y - 31 &= 0 \text{ -----(ii)} \end{aligned}$$

Now for x and y we solve the two equations as the simultaneous equations.

Therefore, by (i) x 2 and (ii) x 1, we get

$$\begin{array}{r} 6x + 4y - 52 = 0 \\ 6x + y - 31 = 0 \\ \hline - \quad - \quad + \\ 3y - 21 = 0 \end{array} \qquad \therefore y = \frac{21}{3} = 7$$

Putting $y = 7$ in eqn (i), we get

$$3x + 2(7) - 26 = 0$$

$$3x - 12 = 0$$

$$x = \frac{12}{3} = 4.$$

Hence $x = 4$ and $y = 7$.

Now to find 'r' we express the equations in the form $y = a + bx$

So, from eqns (i) and (ii)

$$y = -\frac{3}{2}x - \frac{26}{2} \quad \text{and} \quad y = -\frac{6}{1}x + \frac{31}{1}$$

$$\therefore b_1 = -\frac{3}{2} = -1.5 \quad \therefore b_2 = \frac{6}{1} = -6$$

since, $b_1 < b_2$ (i.e. b_1 is smaller in number irrespective of sign + or -)

\therefore Equation (i) is regression of y on x and $b_{yx} = -1.5$

Hence, eqn (ii) is regression of x on y and $b_{xy} = -1/6 = -0.16$

Now we find, $r = \sqrt{b_{xy} \times b_{yx}} \quad r = \sqrt{0.16 \times 1.5} = -0.16$

Note: The sign of 'r' is same as the sign of regression coefficients

Now to find $6x$ when $6y = 3$, we use the formula,

$$b_{yx} = r \frac{\sigma_x}{\sigma_y}$$

$$-1.5 = -\frac{0.16 \times 3}{6x}$$

$$\therefore 6x = \frac{0.48}{1.5} = 0.32$$

Hence means $\bar{x} = 4$, $\bar{y} = 7$, $r = -0.16$ and $6x = 0.32$.

EXERCISES

1. What is mean by Regression? Explain the use of regression in the statistical analysis.
2. Why are there two Regressions? Justify.
3. State the difference between Correlation and Regression.
4. Obtain the two regression equations from the data given bellow.

X: 7 4 6 5 8

Y: 6 5 9 8 2

Hence estimate y when $x = 10$.

5. The data given below are the years of experience (x) and monthly wages (y) for a group of workers. Obtain the two regression equations and approximate the monthly wages of a workers who have completed 15 years of service.

Experience: In years	11	7	9	5	8	6	10
Monthly wages: (in '000Rs.)	10	8	6	8	9	7	11

6. Following results are obtained for a bivariate data. Obtain the two regression equations and find y when $x = 12$

$$n = 15 \quad \Sigma x = 130 \quad \Sigma y = 220 \quad \Sigma x^2 = 2288 \quad \Sigma y^2 = 5506 \quad \Sigma xy = 3467$$

7. Marks scored by a group of 10 students in the subjects of Maths and Stats in a class test are given below. Obtain a suitable regression equation to find the marks of a student in the subject of Stats who have scored 25 marks in Maths.

Student no:	1	2	3	4	5	6	7	8	9	10
Marks in Maths	13	18	9	6	14	10	20	28	21	16
Marks in Stats:	12	25	11	7	16	12	24	25	22	20

8. The data given below are the price and demand for a certain commodity over a period of 7 years. Find the regression equation of Price on Demand and hence obtain the most likely demand for the in the year 2008 when it's price is Rs.23.

Year:	2001	2002	2003	2004	2005	2006	2007
Price(in RS):	15	12	18	22	19	21	25
Demand (100 units)	89	86	90	105	100	110	115

9. For a bivariate data the following results were obtained

$$\bar{x} = 53.2, \quad \bar{y} = 27.9, \quad \sigma_x = 4.8, \quad \sigma_y = \sigma.4 \text{ and } r = 0.75$$

Obtain the two regression equations, find the most probable value of x when $y = 25$.

10. A sample of 50 students in a school gave the following statistics about Marks of students in Subjects of Mathematics and Science,

Subjects:	Mathematics	Science
Mean	58	79
S.D.	12	18

Coefficient of correlation between the marks in Mathematics and marks in Science is 0.8. Obtain the two regression equations and approximate the marks of a student in the subject of Mathematics whose score in Science is 65.

11. It is known that the Advertisement promotes the Sales of the company. The company's previous records give the following results.

Expenditure on Advertisement (Rs. In Lakh)		Sales (Rs. In Lakh)
Mean	15	190
S.D.	6	20

Coefficient of correlation between sales and expenditure on Advertisement is 0.6. Using the regression equation find the likely sales when advertisement budget is Rs.25 Lakh.

12. Find the values of \bar{x} , \bar{y} , and r from the two regression equations given below. $3x+2y-26=0$ and $6x+y-31=0$. Also find σ_x when $\sigma_y = 3$.

13. Two random variables have the regression equations:
 $5x+7y-22=0$ and $6x+2y-20=0$. Find the mean values of x and y . Also find S.D. of x when S.D. of $y = 5$.

14. The two regression equations for a certain data were $y = x+5$ and $16x = 9y-94$. Find values of \bar{x} , \bar{y} and r . Also find the S.D. of y when S.D. of x is 2.4.



Unit-5

TIME SERIES

Unit Structure :

- 5.0 Objectives
- 5.1 Introduction
- 5.2 Importance of Time Series Analysis
- 5.3 Components of Time Series
- 5.4 Methods to find Trend

5.0 OBJECTIVES

From this chapter student should learn analysis of data using various methods. Methods involve moving average method and least square method seasonal fluctuations can be studied by business for casting method.

5.1 INTRODUCTION

Every business venture needs to know their performance in the past and with the help of some predictions based on that, would like to decide their strategy for the present. By studying the past behavior of the characteristics, the nature of variation in the value can be determined. The values in the past can be compared with the present values of comparisons at different places during formulation of future plan and policies. This is applicable to economic policy makers, meteorological department, social scientists, political analysis. Forecasting thus is an important tool in Statistical analysis. The statistical data, particularly in the field of social science, are dynamic in nature. Agricultural and Industrial production increase every year or due to improved medical facilities, there is decline in the death rate over a period of time. There is increase in sales and exports of various products over a period of years. Thus, a distinct change (either increasing or decreasing) can be observed in the value of time-series.

A time series is a sequence of value of a phenomenon arranged in order of their occurrence. Mathematically it can be expressed as a function, namely $y = f(t)$ where t represents time and y represents the corresponding values. That is, the value y_1, y_2, y_3, \dots of a phenomenon with respect to time periods t_1, t_2, t_3, \dots Form a Time Series.

Forecasting techniques facilitates prediction on the basis of a data available from the past. This data from the past is called a time series. A set of observations, of a variable, taken at a regular (fixed and equal) interval of time is called time series. A time series is a bivariate data, with time as the independent variable and the other is the variable under consideration. There are various forecasting method for time series which enable us to study the variation or trends and estimate the same for the future.

5.2 IMPORTANCE OF TIME SERIES ANALYSIS

The analysis of the data in the time series using various forecasting model is called as time analysis. The importance of time series analysis is due to the following reasons:

- *Understanding the past behavior*
- *Planning the future action*
- *Comparative study*

5.3 COMPONENTS OF TIME SERIES

The fluctuation in a time series are due to one or more of the following factors which are called “components” of time series.

(a) Secular Trend :

The general tendency of the data, either to increase, to decrease or to remain constant is called Secular Trend. It is smooth, long term movement of the data. The changes in the values are gradual and continuous. An increasing demand for luxury items like refrigerators or colour T.V. sets reflect increasing trend. The production of steel, cement, vehicles shows a rising trend. On the other hand, decreasing in import of food grains is an example of decreasing trend. The nature of the trend may be linear or curvilinear, in practice, curvilinear trend is more common.

Trend is due to long term tendency. Hence it can be evaluated if the time series is available over a long duration.

(b) Seasonal Variation :

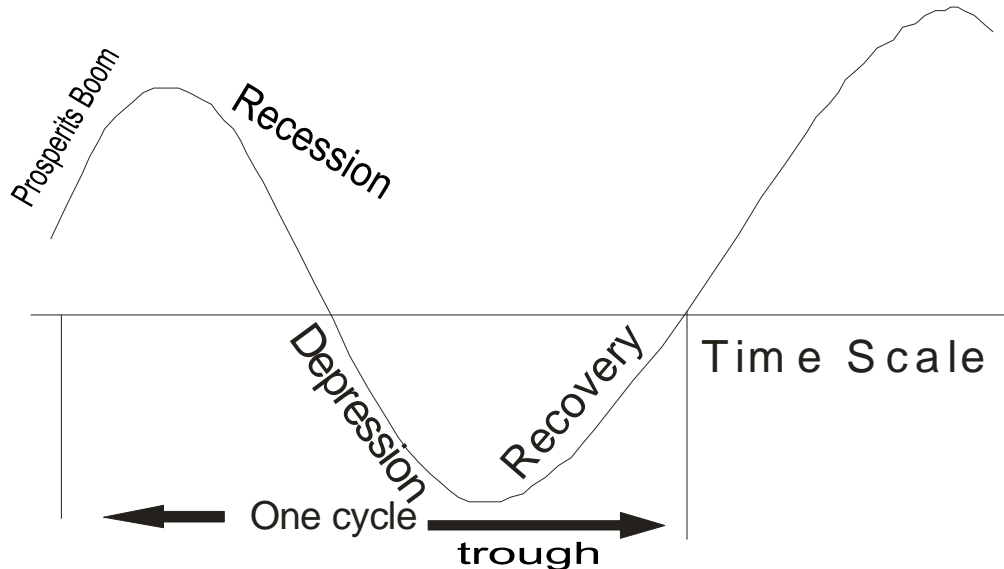
The regular, seasonal change in the time series are called “Seasonal Variation”. It is observed that the demand for umbrellas, raincoats reaches a peak during monsoon or the advertisement of cold drinks, ice creams get a boom in summer. The demand for greeting cards, sweets, increase during festival like Diwali, Christmas. In March, there is maximum withdrawal of bank deposits for adjustment of income-tax payment, so also variation tax-saving schemes shoot up during this period.

The causes, for these seasonal fluctuations, are thus change in weather conditions, the traditions and customs of people etc. The seasonal

component is measured to isolate these change from the trend component and to study their effect, so that, in any business, future production can be planned accordingly and necessary adjustments for seasonal change can be made

(c) Cyclical Variation :

These are changes in time series, occurring over a period which is more than a year. They are recurring and periodic in nature. The period may not be uniform. These fluctuations are due to changes in a business cycle. There are four important phases of any business activity viz. prosperity, recession, depression and recovery. During prosperity, the business flourishes and the profit reaches a maximum level. Thereafter, in recession, the profit decreases, reaching a minimum level during depression. After some time period, the business again recovers (recovery) and it is followed by period of prosperity. The variation in the time series due to these phases in a business cycle are called “Cyclical Variation:.”



The knowledge of cyclic variations is important for a businessman to plan his activity or design his policy for the phase of recession or depression. But one should know that the factors affecting the cyclical variations are quite irregular, difficult to identify and measure. The cyclical variation are denoted by

(d) Irregular Variation :

The changes in the time series which can not be predicated and are erratic in nature are called “Irregular Variation”. Usually, these are short term changes having signification effect on the time series during that time interval. These are caused by unforeseen event like wars, floods, strikes, political charges, etc. During Iran-Iraq war or recent Russian revolution, prices of petrol and petroleum product soared very high. In recent budget, control on capital issued was suddenly removed. As an effect, the all Indian-Index of share market shoted very high, creates all time records.

If the effect of other components of the time series is eliminated, the remaining variation are called “Irregular or Random Variations”. No forecast of these change can be made as they do not reflect any fixed pattern.

MODELS FOR ANALYSIS OF TIME SERIES

The purpose of studying time series is to estimate or forecast the value of the variable. As there are four components of the time series, these are to be studied separately. There are two types of models which are used to express the relationship of the components of the time series. They are additive model and multiplicative model.

O = Original Time Series
 T = Secular Trend
 S = Seasonal Variations
 C = Cyclical Variations and
 I = Irregular Variations

In Additive model, it is assumed that the effect of the individual components can be added to get resultant value of the time series, that is the components are independent of one another. The model can be expressed as

$$O = T + S + C + I$$

In multiplicative model, it is assumed that the multiplication of the individual effect of the components result in the time series, that is, the components are due to different causes but they are not necessarily independent, so that changes in any one of them can affect the other components. This model is more commonly used. It is expressed as

$$O = T \times S \times C \times I$$

If we want to estimate the value in time series, we have to first estimate the four components and then combine them to estimate the value of the time series. The irregular variations can be found. However, we will restrict ourselves, to discuss method of estimating the first components, namely Secular Trend.

5.4 METHODS TO FIND TREND

There are various method to find the trend. The major methods are as mentioned below:

- I. Free Hand Curve.
- II. Method of Semi – Averages.
- III. Method of Moving Averages.
- IV. Method of Least Squares.

we will study only the method of moving average and least squares.

5.4.1.Method of Moving Averages

This is a simple method in which we take the arithmetic average of the given times series over a certain period of time. These average move over period and are hence called as moving averages. The time interval for the average is taken as 3 years, 4 years or 5 years and so on. The average are thus called as 3 yearly, 4 yearly and 5 yearly moving average. The moving averages are useful in smoothing the fluctuations caused to the variable. Obviously larger the time interval of the average more is the smoothing. We shall study the odd yearly (3 and 5) moving average first and then the 4 yearly moving average.

Odd Yearly Moving Average

In this method the total of the value in the time series is taken for the given time interval and is written in front of the middle value. The average so taken is also written in front of this middle value. This average is the trend for that middle year. The process is continued by replacing the first value with the next value in the time series and so on till the trend for the last middle value is calculated. Let us understand this with example:

Example 1:

Find 3 years moving averages and draw these on a graph paper. Also represent the original time series on the graph.

Year	1999	2000	2001	2002	2003	2004	2005	2006	2007
Production (in thousand unit)	12	15	20	18	25	32	30	40	44

Solution:

We calculate arithmetic mean of first three observations viz. 12, 15 and 20, then we delete 12 and consider the next one so that now, average of 15, 20 and 18 is calculated and so on. These averages are placed against the middle year of each group, viz. the year 2000, 2001 and so on. Note moving averages are not obtained for the year 1999 and 2007.

Year	Production (in thousand unit)	3 Years Total	3yrly.Moving Average
1999	12		
2000	15	$12 + 15 + 20 = 47$	$47 / 3 = 15.6$
2001	20	$15 + 20 + 18 = 53$	$53 / 3 = 17.6$
2002	18	$20 + 18 + 25 = 63$	$63 / 3 = 21.0$
2003	25	$18 + 25 + 32 = 75$	$75 / 3 = 25.0$
2004	32	$25 + 32 + 30 = 87$	$87 / 3 = 29.0$
2005	30	$32 + 30 + 40 = 102$	$102 / 3 = 34.0$
2006	40	$30 + 40 + 44 = 114$	$114 / 3 = 38.0$
2007	44		

Example 2:

Find 5 yearly moving average for the following data.

Year	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006
Sales (in lakhs of Rs.)	51	53	56	57	60	55	59	62	68	70

Solution:

We find the average of first five values, namely 51, 53, 56, 57 and 60. Then we omit the first value 51 and consider the average of next five values, that is, 53, 56, 57, 60 and 55. This process is continued till we get the average of the last five values 55, 59, 62, 68 and 70. The following table is prepared.

Year	Sales (in lakhs of Rs.)	5 Years Total	Moving Average (Total / 5)
1997	51
1998	53
1999	56	$51 + 53 + 56 + 57 + 60 = 277$	55.4
2000	57	$53 + 56 + 57 + 60 + 55 = 281$	56.2
2001	60	$56 + 57 + 60 + 55 + 59 = 287$	57.4
2002	55	$57 + 60 + 55 + 59 + 62 = 293$	58.6
2003	59	$60 + 55 + 59 + 62 + 68 = 304$	60.8
2004	62	$55 + 59 + 62 + 68 + 70 = 314$	62.8
2005	68
2006	70

Example 3:

Determine the trend of the following time series using 5 yearly moving averages.

Year	1981	1982	1983	1984	1985	1986	1987	1988	1989	1990	1991
Exports in '000Rs	78	84	80	83	86	89	88	90	94	93	96

Solution: The time series is divided into overlapping groups of five years, their 5 yearly total and average are calculated as shown in the following table.

Year	Export (Y)	5 – yearly total (T)	5 – yearly moving average: (T/5)
1981	78		
1982	84		
1983	80		
1984	83	$78+84+80+83+86 = 411$	$411 / 5 = 82.2$
1985	86	$84+80+83+86+89 = 422$	$422 / 5 = 84.4$
1986	89	$80+83+86+89+88 = 426$	85.2

1987	88	$83+86+89+88+90 = 436$	87.2
1988	90	$86+89+88+90+94 = 447$	89.4
1989	94	$89+88+90+94+93 = 454$	90.8
1990	93	$88+90+94+93+96 = 461$	92.2
1991	96		

Observations:

- I. In case of the 5 – yearly moving average, the total and average for the first two and the last two in the time series is not calculated. Thus, the moving average of the first two and the last two years in the series cannot be computed.
- II. To find the 3 – yearly total (or 5 – yearly total) for a particular years, you can subtract the first value from the previous year's total, and add the next value so as to save your time!

Even yearly moving averages

In case of even yearly moving average the method is slightly different as here we cannot find the middle year of the four years in consideration. Here we find the total for the first four years and place it between the second and the third year value of the variable. These totals are again sunned into group of two, called as centered total and is placed between the two totals. The 4 – yearly moving average is found by dividing these centered totals by 8. Let us understand this method with an example

Example 4: Calculate the 4 yearly moving averages for the following data.

Year	1991	1992	1993	1994	1995	1996	1997	1998	1999
Import in '000Rs	15	18	20	24	21	25	28	26	30

Ans: The table of calculation is show below. Student should leave one line blank after every to place the centered total in between two years.

Years	(Import) Y	4 - Yearly Total	4 - Yearly Centered Total	4 - Yearly Moving Averages :
1991	15	-	-	
1992	18	-	-	
1993	20	77	$77 + 83 = 160$	$160/8 = 20$
1994	24	83	$83 + 90 = 173$	$173/8 = 21.6$
1995	21	90	$90 + 98 = 188$	$188/8 = 23.5$
1996	25	98	$98 + 100 = 198$	$198/8 = 24.8$
1997	28	100	$100 + 109 = 209$	$209/8 = 26.1$
1998	26	109	-	-
1999	30	-	-	-

Example 5:

Find the moving average of length 4 for the following data. Represent the given data and the moving average on a graph paper.

Year	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007
Sales (in thousand unit)	60	69	81	86	78	93	102	107	100	109

Solution: We prepare the following table.

Year	Sale (in thousand unit)	4 Yearly Totals	Centred Total	Moving / Avg. Central = Total / 8
1998	60			
1999	69			
		$60 + 69 + 81 + 86 = 296$		
2000	81		$296 + 314 = 610$	76.25
		$69 + 81 + 86 + 78 = 314$		
2001	86		$314 + 338 = 652$	81.5
		$81 + 86 + 78 + 93 = 338$		
2002	78		$338 + 359 = 697$	87.125
		$86 + 78 + 93 + 102 = 359$		
2003	93		$359 + 380 = 739$	92.375
		$78 + 93 + 102 + 107 = 380$		
2004	102		$380 + 402 = 782$	97.75
		$93 + 102 + 107 + 100 = 402$		
2005	107		$402 + 418 = 820$	102.5
		$102 + 107 + 100 + 109 = 418$		
2006	100			
2007	109			

Note that 4 yearly total are written between the years 1999-2000, 2000-01, 2001-02 etc. and the central total are written against the years 2000, 2001, 2002 etc. so also the moving average are considered w.r.t. years; 2000, 2001 and so on. The moving averages are obtained by dividing the certain total by 8.

The graph of the given set of values and the moving averages against time representing the trend component are shown below. Note that the moving averages are not obtained for the years 1998, 1999, 2006 and 2007. (i.e. first and last two extreme years).

When the values in the time series are plotted, a rough idea about the type of trend whether linear or curvilinear can be obtained. Then, accordingly a linear or second degree equation can be fitted to the values. In this chapter, we will discuss linear trend only.

5.4.2. LEAST SQUARES METHOD:

Let $y = a + bx$ be the equation of the straight line trend where a, b are constant to be determined by solving the following normal equations,

$$\begin{aligned}\sum y &= na + b\sum x \\ \sum xy &= a\sum x + b\sum x^2\end{aligned}$$

where y represents the given time series.

We define x from years such that $\sum x = 0$. So substituting $\sum x = 0$ in the normal equation and simplifying, we get

$$b = \frac{\sum xy}{\sum x^2} \quad \text{and} \quad a = \frac{\sum y}{n}$$

Using the given set of values of the time series, a, b can be calculated and the straight line trend can be determined as $y = a + bx$. This gives the minimum sum of squares line deviations between the original data and the estimated trend values. The method provides estimates of trend values for all the years. The method has mathematical basis and so element of personal bias is not introduced in the calculation. As it is based on all the values, if any values are added, all the calculations are to be done again.

Odd number of years in the time series

When the number of years in the given time series is odd, for the middle year we assume the value of $x = 0$. For the years above the middle year the value given to x are ..., -2, -1 while those after the middle year are values 1, 2, ... and so on.

Even number of years in the time series

When the number of years in the time series is even, then for the upper half the value of x are assumed as ..., -5, -3, -1. For the lower half years, the values of x are assumed as 1, 3, 5, And so on.

Example 6:

Fit a straight line trend for the following data giving the annual profits (in lakhs of Rs.) of a company. Estimate the profit for the year 1999.

Years	1992	1993	1994	1995	1996	1997	1998
Profit	30	34	38	36	39	40	44

Solution: Let $y = a + bx$ be the straight line trend.

The number of years is seven, which is odd. Thus, the value of x is taken as 0 for the middle years 1995, for upper three years as -3, -2, -1 and for lower three years as 1, 2, 3.

The table of computation is as shown below:

Years	Profit (y)	x	xy	x ²	Trend Value: Y _t = a + bx
1992	30	-3	-90	9	31.41
1993	34	-2	-68	4	33.37
1994	38	-1	-38	1	35.33
1995	36	0	0	0	37.29
1996	39	1	39	1	39.25
1997	40	2	80	4	41.21
1998	44	3	132	9	43.17
Total	∑y = 261	∑x = 0	∑xy = 55	∑x ² = 28	

From the table : n = 7, ∑xy = 55, ∑x² = 28, ∑y = 261

$$\text{There fore } b = \frac{\sum xy}{\sum x^2} = \frac{55}{28} = 1.96 \quad \text{and} \quad a = \frac{\sum y}{n} = \frac{261}{7} = 37.29$$

Thus, the straight line trend is $y = 37.29 + 1.96x$.

The trend values in the table for the respective years are calculated by substituting the corresponding value of x in the above trend line equation.

For the trend value for 1992: x = -3:

$$y_{1992} = 37.29 + 1.96(-3) = 37.29 - 5.88 = 31.41$$

Similarly, all the remaining trend values are calculated.

(A short-cut method in case of odd number of years to find the remaining trend values once we calculate the first one, is to add the value of b to the first trend value to get the second trend value, then to the second trend value to get the third one and so on. This is because the difference in the values of x is 1.)

To estimate the profit for the years 1999 in the trend line equation, we substitute the prospective value of x, if the table was extended to 1999. i.e. we put x = 4, the next value after x = 3 for the year 1998.

$$\therefore y_{1999} = 37.29 + 1.96(4) = 45.13$$

There fore the estimated profit for the year 1999 is Rs. 45.13 lakhs.

Example 7:

Fit straight line trend by the method of least squares for the following data representing production in thousand units. Plot the data and the trend line on a graph paper. Hence or otherwise estimate the trend for the years 2007.

Year	1999	2000	2001	2002	2003	2004	2005
Production (in thousand unit)	14	15	17	16	17	20	23

Solution:

Here, the total number of years is 7, an odd number. So we take the center as 1986 the middle-most year and define x as year 2002. The values of x will be -3, -2, -1, 0, 1, 2, 3.

Prepare the following table to calculate the required summations. Note that the trend values can be written in the table only after calculation of a and b .

Year	Production (y)	x	x^2	x y	Trend Values
1999	14	-3	9	-42	13.47
2000	15	-2	4	-30	14.79
2001	17	-1	1	-17	16.11
2002	16	0	0	0	17.43
2003	17	1	1	17	18.75
2004	20	2	4	40	20.07
2005	23	3	3	69	21.39
	122		28	37	

Here, $n = 7$, $\sum y = 122$, $\sum x^2 = 28$, $\sum x y = 37$

Now, a and b are calculated as follows:

$$a = \frac{\sum y}{n} = \frac{122}{7} = 17.4286 \approx 17.43$$

$$b = \frac{\sum xy}{\sum x^2} = \frac{37}{28} = 1.3214 \approx 1.32$$

So, the equation is used to find trend values.

$$y = a + b x$$

$$\text{i.e. } y = 17.43 + 1.32x$$

The equation is used to find trend values.

For the year 1999, $x = -3$, substituting the value of x , we get,

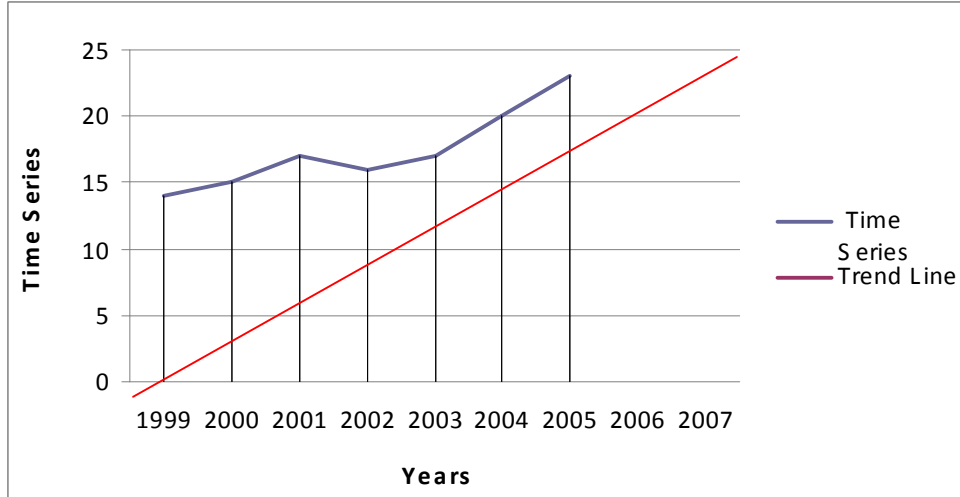
$$y = 17.43 + 1.32(-3) = 17.43 - 3.96 = 13.47$$

to find the remaining trend values we can make use of the property of a straight line that as all the values of x are equidistant with different of one unit (-3, -2, -1, --- and so on), the estimated trend value will also be equidistant with a difference of b unit.

In this case as $b = 1.32$, the remaining trend values for $x = -2, -1, 0, ---$ etc. are obtained by adding $b = 1.32$ to the previous values. So, the trend values are 13.47, 14.79, 16.11, 17.43, 18.75, 20.07 and 21.39.

Now to estimate trend for the year 2007, $x = 5$, substituting in the equation
 $y = 17.43 + 1.32x$
 $= 17.43 + 1.32(5) = 24.03$

So, the estimated trend value for the year 2007 is 24,030 unit.
 For graph of time series, all points are plotted. But for the graph of trend line, any two trend values can be plotted and the line joining these points represents the straight line trend.



For the trend line, the trend values 17.43 and 21.39 for the years 2002 and 2005 are plotted and then a straight line joining these two points is drawn and is extended on both the sides.

The estimate of trend for the year 2007 can also be obtained from the graph by drawing a perpendicular for the year 2007, from x-axis which meet the trend line at point P. From P, a perpendicular on y-axis gives the required trend estimate as 24.

Now, to find straight line trend, when number of years is even, consider the following example.

Example 8:

Fit a straight line trend to the following time –series, representing sales in lakhs of Rs. of a company, for the year 1998 to 2005. Plot the given data well as the trend line on a graph paper. Hence or otherwise estimate trend for the year 2006.

Year	1998	1999	2000	2001	2002	2003	2004	2005
Sales (Lakhs of Rs.)	31	33	30	34	38	40	45	49

Solution:

Here the number of years = 8, an even number, so we define $x = \frac{\text{year} - 2001.5}{0.5}$, so that the values of x are -7, -3, -1, 1, 3, 5 and 7, to get $\sum x = 0$.

Prepare the following table to obtain the summations $\sum x^2$, $\sum y$, $\sum x y$.

Year	Sales (in Lakhs of Rs.)	x	x^2	x y	Trend Values
1998	31	-7	49	-217	28.33
1999	33	-5	25	-165	30.95
2000	30	-3	9	-90	33.57
2001	34	-1	1	-34	36.19
2002	38	1	1	38	38.81
2003	40	3	9	120	41.43
2004	45	5	25	225	44.05
2005	49	7	49	343	46.67
	300		168	220	

Here, $n = 8$, $\sum y = 300$, $\sum x^2 = 168$, $\sum x y = 220$

Now, a and b are calculated as follows:

$$a = \frac{\sum y}{n} = \frac{300}{8} = 37.5$$

$$b = \frac{\sum xy}{\sum x^2} = \frac{220}{168} = 1.31$$

So, the equation of the straight line trend is $y = a + b x$

i.e. $y = 37.5 + 1.31 x$

To obtain the trend values, first calculate y for $x = -7$, for the year 1998

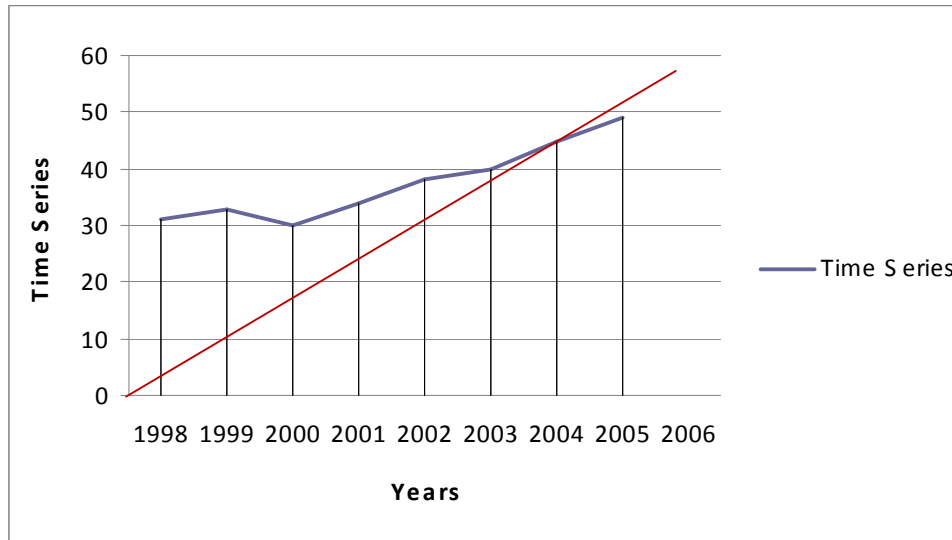
$$\begin{aligned} y &= 37.5 + 1.31 (-7) \\ &= 37.5 - 9.17 = 28.33 \end{aligned}$$

To find the successive trend values, go on addition $2b = 2 \times 1.31 = 2.62$, to the preceding values as in this case the different between x values is of 2 units.

So, the estimated values of trend for $x = -5, -3, -1, 1, 3, 5, 7$ and 7 are 30.95, 33.57, 36.19, 38.81, 41.43, 44.05 and 46.67 respectively. Write down these values in the table.

Hence the estimated trend value for the year 2006 is 49.29 (in lakhs of Rs.).

Now, for the graph of trend line, note that only two trend values 30.95 and 46.67 w.r.t. years 1999 and 2005 are considered as point. The line joining these two points represents trend line.



To estimated the trend for the year 2006, drawn a perpendicular from x-axis at this point meeting the line in P. then from P, draw another perpendicular on y-axis which gives estimate of trend as 49.

Example 9:

Fit a straight line trend to the following data. Draw the graph of the actual time series and the trend line. Estimate the sales for the year 2007.

Year	1998	1999	2000	2001	2002	2003	2004	2005
Sales in '000Rs	120	124	126	130	128	132	138	137

Solution: let $y = a + bx$ be the straight line trend.

The number of years in the given time series is eight, which is an even number. The upper four years are assigned the values of x as 1, 2, 3, and 7. Note that ere the difference between the values of x is 2, but the sum is zero.

Now, the table of computation is completed as shown below:

Years	Profit (y)	X	Xy	X ²	Trend Value: $Y_t = a + bx$
1998	120	-7	-840	49	120.84
1999	124	-5	-620	25	123.28
2000	126	-3	-378	9	125.72
2001	130	-1	-130	1	128.16
2002	128	1	128	1	130.06
2003	132	3	396	9	133.04

2004	138	5	390	25	135.48
2005	137	7	359	49	137.92
Total	$\sum y = 1035$	$\sum x = 0$	$\sum xy = 205$	$\sum x^2 = 168$	

From the table : $n = 8$, $\sum xy = 205$, $\sum x^2 = 168$, $\sum y = 1035$

$$\therefore b = \frac{\sum xy}{\sum x^2} = \frac{205}{168} = 1.22 \quad \text{and} \quad a = \frac{\sum y}{n} = \frac{1035}{8} = 129.38$$

Thus, the straight line trend is $y = 129.38 + 1.22x$.

The trend values in the table for the respective years are calculated by substituting the corresponding value of x in the above trend line equation.

For the trend value for 1998: $x = -7$:

$$y_{1998} = 129.38 + 1.22(-7) = 129.38 - 8.54 = 120.84$$

Similarly, all the remaining trend values are calculated.

(A short-cut method in case of even number of years to find the remaining trend values once we calculate the first one, is to add twice the value of b to the first trend value to get the second trend value, then to the second trend value to get the third one and so on. This is because the difference in the values of x is 2. In this example we add $2 \times 1.22 = 2.44$)

Estimation:

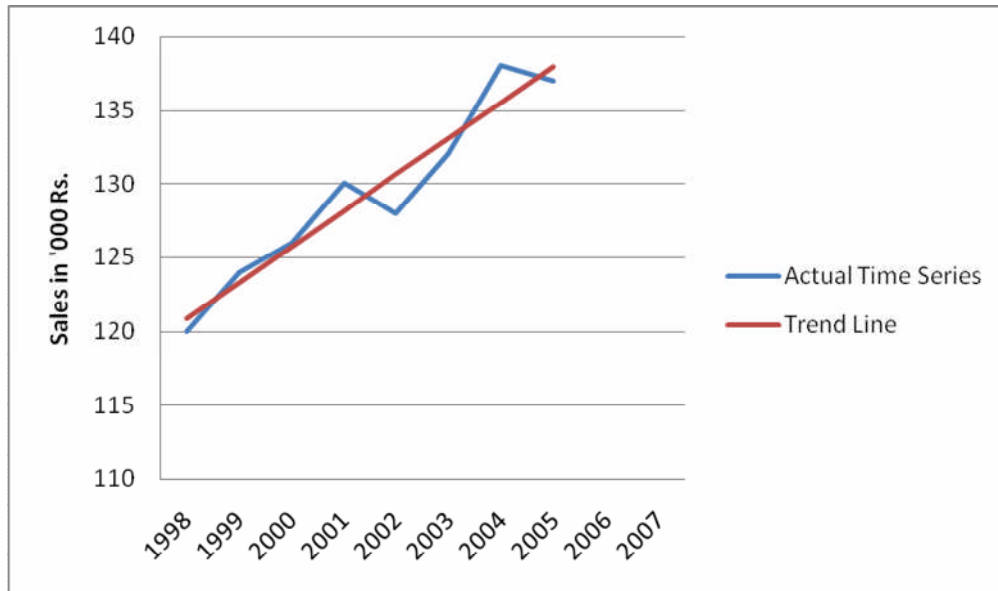
To estimate the profit for the years 2007 in the trend line equation, we substitute the prospective value of x , if the table was extended to 2007. i.e. we put $x = 11$, the next value after $x = 9$ for the year 2006 and $x = 7$ for 2005.

$$y_{2007} = 129.38 + 1.22(11) = 142.8$$

Therefore the estimated profit for the year 2007 is Rs. 1,42,800.

Now we draw the graph of actual time series by plotting the sales against the corresponding year, the period is taken on the X-axis and the sales on the Y-axis. The points are joined by straight lines. To draw the trend line it is enough to plot any two point (usually we take the first and the last trend value) and join it by straight line.

To estimate the trend value for the year 2007, we draw a line parallel to Y-axis from the period 2007 till it meet the trend line at a point say A. From this point we draw a line parallel to the X-axis till it meet the Y-axis at point say B. This point is our estimate value of sales for the year 2007. The graph and its estimate value (graphically) is shown below:



From the graph, the estimated value of the sales for the year 2007 is 142 i.e. Rs 1,42,000 (approximately)

Example 10:

Fit a straight line trend to the following data. Draw the graph of the actual time series and the trend line. Estimate the import for the year 1998.

Year	1991	1992	1993	1994	1995	1996
Import in '000Rs	40	44	48	50	46	52

Solution: Here again the period of years is 6 i.e. even. Proceeding similarly as in the above problem, the table of calculation and the estimation is as follows:

Years	Import (y)	x	xy	x ²	Trend Value: Y _t = a + bx
1991	40	-5	-200	25	41.82
1992	44	-3	-132	9	43.76
1993	48	-1	-48	1	45.7
1994	50	1	50	1	47.64
1995	46	3	138	9	49.58
1996	52	5	260	25	51.52
Total	∑y = 280	∑x = 0	∑xy = 68	∑x ² = 70	

From the table : $n = 6$, $\sum xy = 68$, $\sum x^2 = 70$, $\sum y = 280$

Therefore $b = \frac{\sum xy}{\sum x^2} = \frac{68}{70} = 0.97$ and $\frac{\sum y}{n} = \frac{280}{6} = 46.67$

Thus, the straight line trend is $y = 46.67 + 0.97x$.

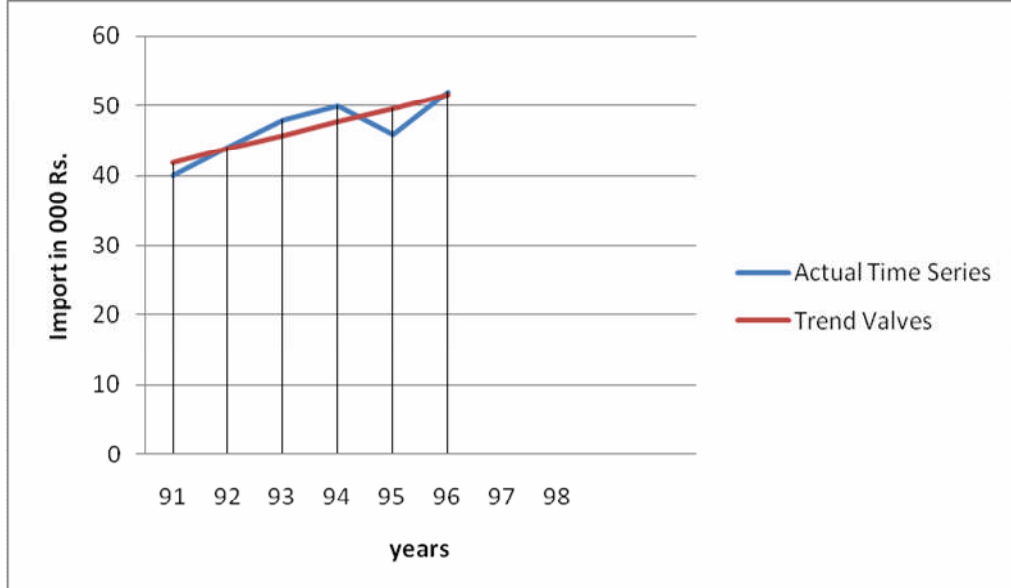
All the remaining trend values are calculated as described in the above problem.

Estimation:

To estimate the import for the year 1998, we put $x = 9$ in the tried line equation. There fore $y_{1997} = 46.67 + 0.97 (9) = 55.4$

There fore the imports for the year 1997 is Rs. 55,400.

The graph of the actual time series and the trend values along with the graphical estimation is an shown below:



From graph the estimated import are Rs. 55,000.

Example 11:

Fit a straight line trend to the following time –series, representing sales in lakhs of Rs. of a company, for the year 1998 to 2005. Plot the given data well as the trend line on a graph paper. Hence or otherwise estimate trend for the year 2006.

Year	1998	1999	2000	2001	2002	2003	2004	2005
Sales (Lakhs of Rs.)	31	33	30	34	38	40	45	49

Solution:

Here the number of years = 8, an even number, so we define

$x = \frac{year - 2001.5}{0.5}$, so that the values of x are -7, -3, -1, 1, 3, 5 and 7, to

get $\sum x = 0$.

Prepare the following table to obtain the summations $\sum x^2$, $\sum y$, $\sum x y$.

Year	Sales (in Lakhs of Rs.)	x	x ²	x y	Trend Values
1998	31	-7	49	-217	28.33
1999	33	-5	25	-165	30.95
2000	30	-3	9	-90	33.57
2001	34	-1	1	-34	36.19
2002	38	1	1	38	38.81
2003	40	3	9	120	41.43
2004	45	5	25	225	44.05
2005	49	7	49	343	46.67
	300		168	220	

Here, $n = 8$, $\sum y = 300$, $\sum x^2 = 168$, $\sum x y = 220$

Now, a and b are calculated as follows:

$$a = \frac{\sum y}{n} = \frac{300}{8} = 37.5$$

$$b = \frac{\sum xy}{\sum x^2} = \frac{220}{168} = 1.31$$

So, the equation of the straight line trend is $y = a + b x$

i.e. $y = 37.5 + 1.31 x$

To obtain the trend values, first calculate y for $x = -7$, for the year 1998

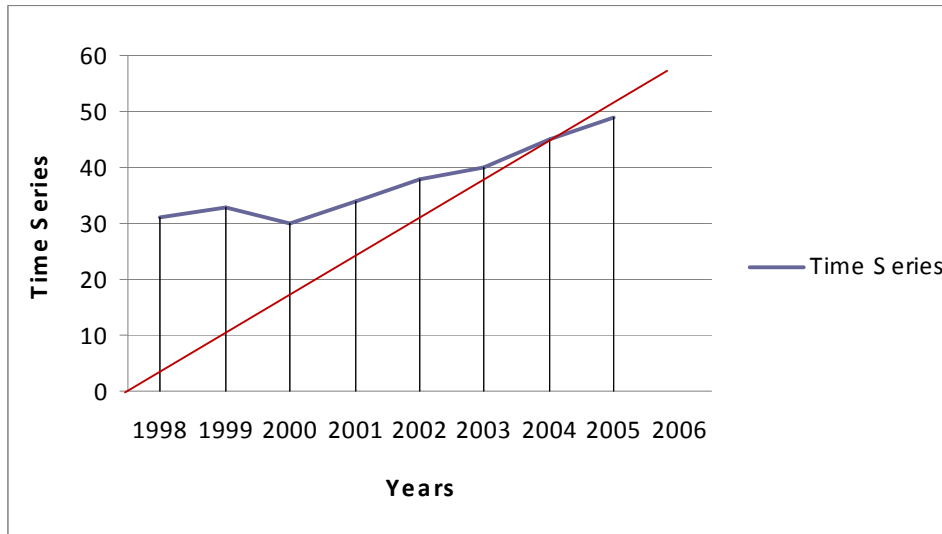
$$\begin{aligned} y &= 37.5 + 1.31 (-7) \\ &= 37.5 - 9.17 = 28.33 \end{aligned}$$

To find the successive trend values, go on addition $2b = 2 \times 1.31 = 2.62$, to the preceding values as in this case the different between x values is of 2 units.

So, the estimated values of trend for $x = -5, -3, -1, 1, 3, 5, 7$ and 7 are 30.95, 33.57, 36.19, 38.81, 41.43, 44.05 and 46.67 respectively. Write down these values in the table.

Hence the estimated trend value for the year 2006 is 49.29 (in lakhs of Rs.).

Now, for the graph of trend line, note that only two trend values 30.95 and 46.67 w.r.t. years 1999 and 2005 are considered as point. The line joining these two points represents trend line.



To estimate the trend for the year 2006, drawn a perpendicular from x-axis at this point meeting the line in P. then from P, draw another perpendicular on y-axis which gives estimate of trend as 49.

MEASUREMENT OF OTHER COMPONENTS

We have studied four method of estimation of Secular Trend. The following procedure is applied to separate the remaining components of the time series.

Using seasonal indices (s), the seasonal variations in a time series can be measured. By removing the trend and the seasonal factors, a combination of cyclical and irregular fluctuations is obtained.

If we assume, multiplicative model, represented by the equation

$$O = T \times S \times C \times I$$

Then, to depersonalize the data, the original time series (O) divided by the seasonal indices (S), which can be express as,

$$\frac{O}{S} = \frac{T \times S \times C \times I}{S} = T \times C \times I$$

If it is further divided by trend values (T), then we have

$$\frac{T \times C \times I}{T} = C \times I$$

Thus a combination of cyclical and irregular variation can be obtained. Irregular fluctuations, because of their nature, can not be eliminated completely, but these can be minimized by taking short term averages and then the estimate of cyclical variation can be obtained.

METHODS TO ESTIMATE SEASONAL FLUCTUATIONS

We have seen methods to separate the trend component of Time Series. Now, let us see, how to separate the seasonal component of it.

Methods of Seasonal Index

It is used to find the effect of seasonal variations in a Time Series. The steps are as follows:

- i. Find the totals for each season, as well as the grand total, say G.
- ii. Find the arithmetic means of these total, and the grand total by dividing the values added.
- iii. Find seasonal indices, representing the seasonal component for each season, using the formula

$$\text{Seasonal Index} = \frac{\text{Average for Seasonal} \times 100}{\text{Grand Average}}$$

$$\text{Where, Grand Average} = \frac{G}{\text{Total No. of Values}}$$

Example 12:

Find the seasonal component of the time series, using method of seasonal indices.

Seasonal / Years	I	II	IV	Grand
2003	33	37	32	31
2004	35	40	36	35
2005	34	38	34	32
2006	36	41	35	36
2007	34	39	35	32

Solution:

	I	II	III	IV	Grand
Total	172	195	172	166	705 (G)
Average	34.4 (172 / 5)	39	34.4	33.2	35.25 (G/20)
Seasonal Index	$\frac{34.4 \times 100}{35.25}$ = 97.59	$\frac{39 \times 100}{35.25}$ = 110.64	$\frac{34.4 \times 100}{35.25}$ = 97.59	$\frac{33.2 \times 100}{35.25}$ = 94.18	

The time series can be deseasonalised by removing the effect of seasonal component from it. It is done using the formula.

$$\text{Deseasonalised Value} = \frac{\text{Original Value} \times 100}{\text{Seasonal Index}}$$

BUSINESS FORECASTING:

In this chapter, few methods of analyzing the past data and predicting the future values are already discussed. Analysis of time series an important role in Business Forecasting. One of the aspects of it estimating future trend values. Now-a-days, any business or industry is governed by factors like supply of raw material, distribution network, availability of land, labour and capital and facilitates like regular supply of power, coal, water, etc. a business has to sustain intricate government regulations, status, everchanging tastes and fashions, the latest technology, cut throat competition by other manufacturers and many other.

While making a forecast, combined effect of above factors should be considered. Scientific method are used to analyse the past business condition. The study reveals the pattern followed by the business in the past. It also bring out the relationship and interdependence of different industries which helps in interpretation of changes in the right perspective. The analysis gives an idea about the components of the time series and their movement in the past. Various indices such as index of production, prices, bank deposits, money rates, foreign exchange position etc. can provide information about short and long term variations, the general trend, the ups downs in a business.

The study of the past data and the comparison of the estimated and actual values helps in pinpointing the areas of shortcoming which can be overcome. For successful business forecasting co-ordination of all departments such as production, sales, marketing is sine-qua-nin, which result in achieving ultimate corporate goals.

There are different theories of Business Forecasting such as

- i. Time lag or Sequence Theory
- ii. Action and Reaction Theory
- iii. Cross Cut Analysis Theory
- iv. Specific Historical Analogy Theory

Of these, Time lag or Sequence Theory is most important. It is based on the fact that there is a time lag between the effect of changes at different stages but there is a sequence followed by these effect e.g. In 80's, the invention of silicon chips brought fourth and fifth generation computers in use. The computers were introduced in various fields such as front-line and back house banking, airlines and railways reservation, new communication technique, home appliances like washing machine etc. this, in turn, increase the demand for qualified personnel in electronic filed to manufacture, handle and maintain these sophisticated machine. It has result in mad rush for admission to various branches of electronics and computer engineering in the recent past.

By applying any one of the these forecasting theories, business forecasting can be made. It should be noted that while collecting the data for analysis, utmost care has to be taken so as to increase the reliability of estimates. The information should be collected by expert investigators, over a long period of time. Otherwise, it may lead to wrong conclusions.

EXERCISE

1. What is a time series ? Describe the various components of a time series with suitable example.
2. What are seasonal variation ? Explain briefly with example.
3. Describe the secular trend component of a time series,

4. What are the method of determining trend in a time series?
5. Compare method of moving average and least squares of estimating trend component.
6. Find the trend values using the method of semi-averages for the following data expressing production in thousand unit of a company for 7 years.
7. Explain the method to calculate 3 yearly and 4 yearly moving averages.
8. What are the merits and demerits of the method of moving average?
9. Explain the simple average method to find the seasonal indices of a time series
10. Calculate trend by considering three yearly moving average for the following time series of price indices for the years 2000-2007. Also plot on the graph the trend values.

Year	2000	2001	2002	2003	2004	2005	2006	2007
Price Index	111	115	116	118	119	120	122	124

11. Determine the trend for the following data using 3 yearly moving averages. Plot the graph of actual time series and the trend values.

Year	1989	1990	1991	1992	1993	1994	1995	1996	1997
Sales in '000Rs	24	28	30	33	34	36	35	40	44

12. Determine the trend for the following data using 3 yearly moving averages. Plot the graph of actual time series and the trend values.

Year	1977	1978	1979	1980	1981	1982	1983	1984
Sales in '000Rs	46	54	52	56	58	62	59	63

13. Determine the trend for the following data using 3 yearly moving averages. Plot the graph of actual time series and the trend values.

14.

Year	1979	1980	1981	1982	1983	1984	1985	1986
Profit in lakhs of Rs	98	100	97	101	107	110	102	105

15. Determine the trend for the following data using 5 yearly moving averages. Plot the graph of actual time series and the trend values.

Year	1980	1982	1984	1986	1988	1990	1992	1994	1996	1998	2000
Values	34	37	35	38	37	40	43	42	48	50	52

16. Determine the trend for the following data giving the production of steel in million tons, using 5 yearly moving averages. Plot the graph of actual time series and the trend values.

Year	1972	1973	1974	1975	1976	1977	1978	1979	1980	1981	1982
Production	28	30.5	32	36.8	38	36	39.4	40.6	42	45	43.5

17. Find five-yearly moving average for the following data which represents production in thousand unit of a small scale industry. Plot the given data as well as the moving average on a graph paper.

Year	1980	1981	1982	1983	1984	1985	1986	1987	1988	1989	1990
Production	110	104	78	105	109	120	115	110	115	122	130

Ans. The trend values are 101.2, 103.2, 105.4, 111.8, 113.8, 116.4 and 118.4 for the years 1982 to 1988.

18. Find the trend component of the following time series of production in thousand kilogram during 1971-1980. Plot the moving average and the original time on a graph paper.

Year	1971	1972	1973	1974	1975	1976	1977	1978	1979	1980
Production	12	15	18	17	16	20	23	22	24	25

Ans. The trend values are 16, 17.125, 18.375, 19.625, 21.25, 22.875 for the years 1973 to 1978.

19. Fit a straight line trend to the following data representing import in million Rs. of a certain company. Also find an estimate for the year 2008.

Year	2000	2001	2002	2003	2004	2005	2006
Import	48	50	58	52	45	41	49

Ans. The straight line trend is $y = 49 - x$. the trend values are 52, 51, 50, 49, 48, 47 and 46 respectively and the estimate trend for the year 2006 is 44 million Rs.

20. The production of a certain brand of television sets in thousand unit is given below. Fit a straight line trend to the data. Plot the given data and the trend line on graph find an estimate for the year 2004.

Year	1997	1998	1999	2000	2001	2002	2003
production	865	882	910	925	965	1000	1080

Ans. The straight line trend is $y = 947.71 + 33.43 x$. the trend values are 846.42, 879.85, 913.28, 946.71, 1013.57 and 1047. The estimate for the year 2004 is 1080.43 thermal million.

21. The straight line trend by the method of least squares for the following data which represents the expenditure in lakhs od Rs. on advertisement of

a certain company. Also find an estimate for the year 2005. Plot the given data and the trend line on a graph paper.

Year	1997	1998	1999	2000	2001	2002	2003	2004
Expenditure	21	24	32	40	38	49	57	60

Ans. The trend is $y = 40.13 + 2.9x$. the trend values are 19.83, 25.62, 31.43, 37.23, 43.03, 48.83, 54.63 and 60.43, 2005 is 66.23.

22. Use the method of least squares to find straight line trend for the following time series of production in thousand units 1981 – 1988. Also estimate trend for the year 2003.

Year	1995	1996	1997	1998	1999	2000	2001	2002
Production	80	90	92	83	94	99	92	102

Ans. The straight line trend is $y = 91.5 + 1.167 x$. the trend values are 83.331, 85.665, 87.999, 90.333, 92.667, 95.001, 97.335 and 99.669. the estimate of trend, for the year 2003 is 102.003

23. Calculate seasonal indices for the following data:

Year	I	II	III	IV
2003	55	53	57	51
2004	56	55	60	53
2005	57	56	61	54

Ans. 100.59, 98.2, 106.57, 94.61

24. Determine the trend for the following data giving the production of wheat in thousand tons from the years 1980 to 1990, using the 5-yearly moving averages. Plot the graph of actual time series and the trend values.

Year	1980	1981	1982	1983	1984	1985	1986	1987	1988	1989	1990
Production	13.5	14.7	17	16.2	18.1	20.4	22	21.2	24	25	26.6

25. Determine the trend for the following data giving the income (in million dollars) from the export of a product from the year 1988 to 1999. Use the 4-yearly moving average method and plot the graph of actual time series and trend values.

Year	1988	1989	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999
Income	340	360	385	470	430	444	452	473	490	534	541	576

26. Using the 4-yearly moving average method find the trend for the following data.

Year	1967	1968	1969	1970	1971	1972	1973	1974	1975	1976	1977
Value	102	100	103	105	104	109	112	115	113	119	117

27. Determine the trend for the following data giving the sales (in '00 Rs.) of a product per week for 20 weeks. Use appropriate moving average method.

Week	1	2	3	4	5	6	7	8	9	10
Sales	22	26	28	25	30	35	39	36	30	32
Week	11	12	13	14	15	16	17	18	19	20
Sales	29	34	36	35	35	39	43	48	52	49

28. An online marketing company works 5-days a week. The day-to-day total sales (in '000 Rs) of their product for 4 weeks are given below. Using a proper moving average method find the trend values.

Days	1	2	3	4	5	6	7	8	9	10
Sales	12	16	20	17	18	20	26	25	27	30
Days	11	12	13	14	15	16	17	18	19	20
Sales	35	32	32	38	36	35	34	38	40	41

29. Fit a straight line trend to the following data. Draw the graph of the actual time series and the trend line. Estimate the sales for the years 2000.

Year	1989	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999
Sales in '000 Rs	45	47	49	48	54	58	53	59	62	60	64

30. Fit a straight line trend to the following data. Draw the graph of the actual time series and the trend line. Estimate the sales for the years 2001.

Year	1989	1990	1991	1992	1993	1994	1995	1996	1997	1998
Profit in '000 Rs	76	79	82	84	81	84	89	92	88	90

31. Fit a straight line trend to the following data. Draw the graph of the actual time series and the trend line. Estimate the sales for the years 2007.

Year	1998	1999	2001	2002	2003	2004	2005	2006
Profit in '000 Rs	116	124	143	135	138	146	142	152

32. Fit a straight line trend to the following data giving the number of casualties (in hundred) of motorcyclists without helmet. Estimate the number for the year 1999.

Year	1992	1993	1994	1995	1996	1997	1998
No of casualties	12	14.2	15.2	16	18.8	19.6	22.1

33. Fit a straight line trend to the following data. Draw the graph of the actual time series and the trend line. Estimate the import for the years 2002

Year	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
Import in'000 Rs.	55	52	50	53	54	56	58	60	57	59

34. Fit a straight line trend to the following data giving the price of crude oil per barrel in USD. Draw the graph of the actual time series and the trend line. Estimate the sales for the year 2003.

Year	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001
Price per barrel	98	102	104.5	108	105	109	112	118	115	120

35. Apply the method of least squares to find the number of student attending the library in the month of May of the academic year 2005 – 2006 from the following data.

Month	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Jan	Feb	Mar	Apr
Students	105	120	160	225	180	115	124	138	176	230	180

36. Assuming that the trend is absent, find the seasonal indices for the following data and also find the deseasonalized values.

Quarters	I	II	III	IV
1977	10	12	14	16
1978	12	15	18	22
1979	16	18	20	24
1980	24	26	28	34

37. Calculate seasonal indices for the following data:

Year	I	II	III	IV
2003	55	53	57	51
2004	56	55	60	53
2005	57	56	61	54

Ans. 100.59, 98.2, 106.57, 94.61



Unit-6

INDEX NUMBERS

Unit Structure :

- 6.0 Objectives
- 6.1 Introductions
- 6.2 Importance of Index Numbers
- 6.3 Price Index Numbers
- 6.4 Cost of Living Index Number or Consumer Price Index Number
- 6.5 Use of Cost of Living Index Numbers
- 6.6 Real Income
- 6.7 Demerits of Index Numbers

6.0 OBJECTIVES

- To understand about the importance of Index Numbers.
- To understand different types of Index Numbers and their computations.
- To understand about *Real Income* and *Cost of Living Index Numbers*.
- To understand the problems in constructing Index Numbers.
- To know the merits and demerits of Index Numbers.

6.1 INTRODUCTION

Every variable undergoes some changes over a period of time or in different regions or due to some factors affecting it. These changes are needed to be measured. In the last chapter we have seen how a time series helps in estimating the value of a variable in future. But the magnitude of the changes or variations of a variable, if known, are useful for many more reasons. For example, if the changes in prices of various household commodities are known, one can plan for a proper budget for them in advance. If a share broker is aware of the magnitude of fluctuations in the price of a particular share or about the trend of the market he can plan his course of action of buying or selling his shares. Thus, we can feel that there is a need of such a measure to describe the changes in prices, sales, profits, imports, exports etc, which are useful from a common man to a business organization.

Index number is an important statistical relative tool to measure the changes in a variable or group of variables with respect to time, geographical conditions and other characteristics of the variable(s). Index number is a relative measure, as it is independent of the units of the variable(s) taken in to consideration. This is the advantage of index numbers over normal averages. All the averages which we studied before

are absolute measures, *i.e.* they are expressed in units, while index numbers are percentage values which are independent of the units of the variable(s). In calculating an index number, a base period is considered for comparison and the changes in a variable are measured using various methods.

Though index numbers were initially used for measuring the changes in prices of certain variables, now it is used in almost every field of physical sciences, social sciences, government departments, economic bodies and business organizations. The gross national product (GNP), per capita income, cost of living index, production index, consumption, profit/loss etc every variable in economics uses this as a tool to measure the variations. Thus, the fluctuations, small or big, in the economy are measured by index numbers. Hence it is called as a barometer of economics.

6.2 IMPORTANCE OF INDEX NUMBERS

The important characteristics of Index numbers are as follows:

(1) ***It is a relative measure:*** As discussed earlier index numbers are independent of the units of the variable(s), hence it a special kind of average which can be used to compare different types of data expressed in different units at different points of time.

(2) ***Economical Barometer:*** A barometer is an instrument which measures the atmospheric pressure. As the index numbers measure all the ups and downs in the economy they are hence called as the economic barometers.

(3) ***To generalize the characteristics of a group:*** Many a time it is difficult to measure the changes in a variable in complete sense. For example, it is not possible to directly measure the changes in a business activity in a country. But instead if we measure the changes in the factors affecting the business activity, we can generalize it to the complete activity. Similarly the industrial production or the agricultural output cannot be measured directly.

(4) ***To forecast trends:*** Index numbers prove to be very useful in identifying trends in a variable over a period of time and hence are used to forecast the future trends.

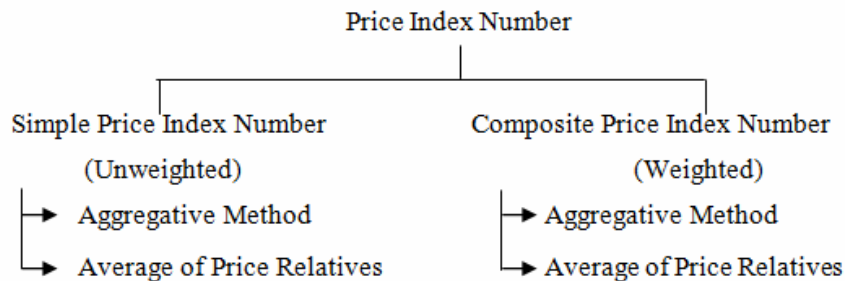
(5) ***To facilitate decision making:*** Future estimations are always used for long term and short term planning and formulating a policy for the future by government and private organizations. Price Index numbers provide the requisite for such policy decisions in economics.

(6) ***To measure the purchasing power of money and useful in deflating:*** Index numbers help in deciding the actual purchasing power of money. We often hear from our elders saying that “ In our times the salary was just Rs. 100 a month and you are paid Rs. 10,000, still you are not happy!”

The answer is simple (because of index numbers!) that the money value of Rs. 100, 30 years before and now is drastically different. Calculation of *real income* using index numbers is an important tool to measure the actual income of an individual. This is called as deflation.

There are different types of index numbers based on their requirement like, price index, quantity index, value index etc. The price index is again classified as single price index and composite price index.

6.3 PRICE INDEX NUMBERS



The price index numbers are classified as shown in the following diagram:

Notations:

P_0 : Price in Base Year

Q_0 : Quantity in Base Year

P_1 : Price in Current Year

Q_1 : Quantity in Current Year

The *suffix '0'* stands for the *base year* and the *suffix '1'* stands for the *current year*.

6.3.1 Simple (Unweighted) Price Index Number By Aggregative Method

In this method we define the price index number as the ratio of sum of prices in current year to sum of prices in base year and express it in percentage. *i.e.* multiply the quotient by 100.

Symbolically,
$$I = \frac{\sum P_1}{\sum P_0} \times 100 \quad \dots (1)$$

Steps for computation:

1. The total of all base year prices is calculated and denoted by $\sum P_0$.
2. The total of all current year prices is calculated and denoted by $\sum P_1$.
3. Using the above formula, simple price index number is computed.

Example 1

For the following data, construct the price index number by simple aggregative method:

Commodity	Unit	Price in	
		1985	1986
A	Kg	10	12
B	Kg	4	7
C	Litre	6	7
D	Litre	8	10

Solution: Following the steps for computing the index number, we find the totals of the 3rd and 4th columns as shown below:

Commodity	Unit	Price in	
		1985(P_0)	1986(P_1)
A	Kg	10	12
B	Kg	4	7
C	Litre	6	7
D	Litre	8	10

$$\therefore I = \frac{\Sigma P_1}{\Sigma P_0} \times 100 = \frac{36}{28} \times 100 = 128.57$$

Meaning of the value of I:

$I = 128.57$ means that the prices in 1986, as compared with that in 1985 have increased by 28.57 %.

6.3.2 Simple (Unweighted) Price Index Number by Average of Price Relatives Method

In this method the price index is calculated for every commodity and its arithmetic mean is taken. *i.e.* the sum of all price relative is divided by the total number of commodities.

Symbolically, if there are n commodities in to consideration, then the simple price index number of the group is calculated by the formula:

$$I = \frac{1}{n} \Sigma \left(\frac{P_1}{P_0} \times 100 \right) \quad \dots (2)$$

Steps for computation

1. The price relatives for each commodity are calculated by the formula:

$$\frac{P_1}{P_0} \times 100.$$

2. The total of these price relatives is calculated and denoted as:

$$\Sigma \left(\frac{P_1}{P_0} \times 100 \right).$$

3. The arithmetic mean of the price relatives using the above formula no.

(2) gives the required price index number.

Example 2

Construct the simple price index number for the following data using average of price relatives method:

Commodity	Unit	Price in	
		1997	1998
Rice	Kg	10	13
Wheat	Kg	6	8
Milk	Litre	8	10
Oil	Litre	15	18

Solution: In this method we have to find price relatives for every commodity and then total these price relatives. Following the steps for computing as mentioned above, we introduce first, the column of price relatives. The table of computation is as follows:

Commodity	Unit	Price in		$\frac{P_1}{P_0} \times 100$
		1997(P_0)	1998(P_1)	
Rice	Kg	10	13	130
Wheat	Kg	6	8	133.33
Milk	Litre	8	10	125
Oil	Litre	15	18	120
Total:				508.33

Now, $n = 4$ and the total of price relatives is 508.33

$$\therefore I = \frac{1}{n} \sum \left(\frac{P_1}{P_0} \times 100 \right) = \frac{508.33}{4} = 127.08$$

The prices in 1998 have increased by 27 % as compared with in 1997.

Remark:

1. The simple aggregative method is calculated without taking into consideration the units of individual items in the group. This may give a misleading index number.
2. This problem is overcome in the average of price relatives method, as the individual price relatives are computed first and then their average is taken.
3. Both the methods are unreliable as they give equal weightage to all items in consideration which is not true practically.

6.3.3 Weighted Index Numbers by Aggregative Method

In this method weights assigned to various items are considered in the calculations. The products of the prices with the corresponding weights are computed; their totals are divided and expressed in percentages.

Symbolically, if W denotes the weights assigned and P_0, P_1 have their usual meaning, then the weighted index number using aggregative method is given by the formula:

$$I = \frac{\Sigma P_1 W}{\Sigma P_0 W} \times 100 \quad \dots (3)$$

Steps to find weighted index number using aggregative method

1. The columns of P_1W and P_0W are introduced.
2. The totals of these columns are computed.
3. The formula no. (3) is used for computing the required index number.

Example 3

From the following data, construct the weighted price index number:

Commodity	A	B	C	D
Price in 1982	6	10	4	18
Price in 1983	9	18	6	26
Weight	35	30	20	15

Solution: Following the steps mentioned above, the table of computations is as follows:

Commodity	Weight (W)	Price in 1982 (P_0)	P_0W	Price in 1983 (P_1)	P_1W
A	35	6	210	9	315
B	30	10	300	18	540
C	20	4	80	6	120
D	15	18	270	26	390
Total	-	-	$\Sigma P_0W = 860$	-	$\Sigma P_1W = 1365$

Using the totals from the table, we have

$$\text{Weighted Index Number } I = \frac{\Sigma P_1 W}{\Sigma P_0 W} \times 100 = \frac{1365}{860} \times 100 = 158.72$$

Remark:

There are different formulae based on what to be taken as the weight while calculating the weighted index numbers. Based on the choice of the weight we are going to study here three types of weighted index numbers: (1) Laspeyre's Index Number, (2) Paasche's Index Number and (3) Fisher's Index Number.

(1) Laspeyre's Index Number:

In this method, Laspeyre assumed the *base quantity* (Q_0) as the *weight* in constructing the index number. Symbolically, P_0 , P_1 and Q_0 having their usual meaning, the Laspeyre's index number denoted by I_L is

given by the formula:

$$I_L = \frac{\Sigma P_1 Q_0}{\Sigma P_0 Q_0} \times 100$$

... (4)

Steps to compute I_L :

1. The columns of the products $P_0 Q_0$ and $P_1 Q_0$ are introduced.
2. The totals of these columns are computed.
3. Using the above formula no. (4), I_L is computed.

Example 4

From the data given below, construct the Laspeyre's index number:

Commodity	1965		1966
	Price	Quantity	Price
A	5	12	7
B	7	12	9
C	10	15	15
D	18	5	20

Solution: Introducing the columns of the products $P_0 Q_0$ and $P_1 Q_0$, the table of computation is completed as shown below:

Commodity	1965		1966	$P_0 Q_0$	$P_1 Q_0$
	Price (P_0)	Quantity (Q_0)	Price (P_1)		
A	5	12	7	60	84
B	7	12	9	84	108
C	10	15	15	150	225
D	18	5	20	90	100
Total	-	-	-	$\Sigma P_0 Q_0 = 384$	$\Sigma P_1 Q_0 = 517$

Using the totals from the table and substituting in the formula no. (4), we have

$$I_L = \frac{\Sigma P_1 Q_0}{\Sigma P_0 Q_0} \times 100 = \frac{517}{384} \times 100 = 134.64$$

(2) Paasche's Index Number:

In this method, Paasch assumed the current year quantity (Q_1) as the weight for constructing the index number. Symbolically, P_0 , P_1 and Q_1 having their usual meaning, the Paasche's index number denoted by I_P is given by the formula:

$$I_P = \frac{\Sigma P_1 Q_1}{\Sigma P_0 Q_1} \times 100$$

... (5)

The steps for computing I_P are similar to that of I_L .

Example 5

From the data given below, construct the Paasche's index number:

Commodity	1985		1986
	Price	Price	Quantity
A	5	8	10
B	10	14	20
C	6	9	25
D	8	10	10

Solution: Introducing the columns of the products P_0Q_1 and P_1Q_1 , the table of computations is completed as shown below:

Commodity	1985		1986	P_0Q_1	P_1Q_1
	Price (P_0)	Price (P_1)	Quantity (Q_1)		
A	5	8	10	50	80
B	10	14	20	200	280
C	6	9	25	150	225
D	8	10	10	80	100
Total	-	-	-	ΣP_0Q_1 = 480	ΣP_1Q_1 = 685

Using the totals from the table and substituting in the formula no. (5), we

$$\text{have: } I_P = \frac{\Sigma P_1Q_1}{\Sigma P_0Q_1} \times 100 = \frac{685}{480} \times 100 = 142.71$$

(3) Fisher's Index Number:

Fisher developed his own method by using the formulae of Laspeyre and Paasche. He defined the index number as the geometric mean of I_L and I_P . Symbolically, the Fisher's Index number denoted as I_F is given by the

$$\text{formula: } I_F = \sqrt{I_L \times I_P} = \sqrt{\frac{\Sigma P_1Q_0}{\Sigma P_0Q_0} \times \frac{\Sigma P_1Q_1}{\Sigma P_0Q_1}} \times 100 \quad \dots (6)$$

Note:

1. The multiple 100 is outside the square root sign.
2. While computing products of the terms, care should be taken to multiply corresponding numbers properly.

Example 6

From the following data given below, construct the (i) Laspeyre's index number, (ii) Paasche's index number and hence (iii) Fisher's index number.

Item	1975		1976	
	Price	Quantity	Price	Quantity
A	4	12	6	16
B	2	16	3	20
C	8	9	11	14

Solution: Introducing four columns of the products of P_0Q_0 , P_0Q_1 , P_1Q_0 and P_1Q_1 , the table of computations is completed as shown below:

Item	P_0	Q_0	P_1	Q_1	P_0Q_0	P_0Q_1	P_1Q_0	P_1Q_1
A	4	12	6	16	48	64	72	96
B	2	16	3	20	32	40	48	60
C	8	9	11	14	72	112	99	154
Total					152	216	219	310

From the table, we have $\Sigma P_0Q_0 = 152$, $\Sigma P_0Q_1 = 216$, $\Sigma P_1Q_0 = 219$ and $\Sigma P_1Q_1 = 310$

$$\therefore I_L = \frac{\Sigma P_1Q_0}{\Sigma P_0Q_0} \times 100 = \frac{219}{152} \times 100 = 144.08$$

$$\therefore I_P = \frac{\Sigma P_1Q_1}{\Sigma P_0Q_1} \times 100 = \frac{310}{216} \times 100 = 143.52$$

$$\therefore I_F = \sqrt{I_L \times I_P} = \sqrt{144.08 \times 143.52} = 143.8$$

Remark:

1. Laspeyre's index number though popular has a drawback that it does not consider the change in consumption over a period. (as it does not take into account the current quantity).
2. Paasche's index number overcomes this by assigning the current year quantity as weight.
3. Fisher's index number being the geometric mean of both these index numbers, it considers both the quantities. Hence it is called as the ideal index number.

Example 7

From the following data given below, construct the Kelly's index number:

Item	Base Year		Current Year	
	Price	Quantity	Price	Quantity
A	18	20	24	22
B	9	10	13	16
C	10	15	12	19
D	6	13	8	15
E	32	14	38	18

Solution: Introducing the columns of $Q = \frac{Q_0 + Q_1}{2}$, P_0Q and P_1Q , the table of computations is completed as shown below:

Item	Q_0	Q_1	Q	P_0	P_0Q	P_1	P_1Q
A	20	22	21	18	378	24	504
B	10	16	13	9	117	13	169
C	15	19	17	10	170	12	204
D	13	15	14	6	84	8	112
E	14	18	16	32	512	38	608
Total					1261	--	1597

From the table, we have $\sum P_0Q = 1261$ and $\sum P_1Q = 1597$

$$\therefore I_K = \frac{\sum P_1Q}{\sum P_0Q} \times 100 = \frac{1597}{1261} \times 100 = 126.65$$

6.3.4 Weighted Index Numbers using average of price relatives method

This is similar to what we have seen in subsection 7.3.2. Here the individual price relatives are computed first. These are multiplied with the corresponding weights. The ratio of the sum of the products and the total value of the weight is defined to be the weighted index number.

Symbolically, if W denotes the weights and I denote the price relatives

then the weighted index number is given by the formula: $\frac{\sum IW}{\sum W} \dots$ (8)

One of the important weighted index number is the *cost of living index number*, also known as the *consumer price index (CPI) number*.

6.4 COST OF LIVING INDEX NUMBER OR CONSUMER PRICE INDEX NUMBER

There are two methods for constructing this index number:

(1) Aggregative expenditure method and (2) Family Budget Method

(1) In aggregative expenditure method we construct the index number by taking the base year quantity as the weight. In fact this index number is nothing but the Laspeyre's index number.

(2) In family budget method, value weights are computed for each item in the group and the index number is computed using the formula:

$$\frac{\sum IW}{\sum W}, \text{ where } I = \frac{P_1}{P_0} \times 100 \text{ and } W = P_0Q_0 \dots (9)$$

Example 8

A survey of families in a city revealed the following information:

Item	Food	Clothing	Fuel	House Rent	Misc.
% Expenditure	30	20	15	20	15
Price in 1987	320	140	100	250	300
Price in 1988	400	150	125	250	320

What is the cost of living index number for 1988 as compared to that of 1987?

Solution: Here % expenditure is taken as the weight (W). The table of computations is as shown below:

Item	P_0	P_1	$I = \frac{P_1}{P_0} \times 100$	% Expenditure (W)	IW
Food	320	400	125	30	3750
Clothing	140	150	107.14	20	2142.8
Fuel	100	125	125	15	1875
House Rent	250	250	100	20	2000
Miscellaneous	300	320	106.67	15	1600.05
Total				$\Sigma W = 100$	11367.85

From the table, we have $\Sigma W = 100$ and $\Sigma IW = 11376.85$

$$\therefore \text{cost of living index number} = \frac{\Sigma IW}{\Sigma W} = \frac{11367.85}{100} = 113.68$$

6.5 USE OF COST OF LIVING INDEX NUMBERS

1. These index numbers reflect the effect of rise and fall in the economy or change in prices over the standard of living of the people.
2. These index numbers help in determining the purchasing power of money which is the reciprocal of the cost of living index number.
3. It is used in deflation. *i.e.* determining the actual income of an individual. Hence it also used by the management of government or private organizations to formulate their policies regarding the wages, allowance to their employees.

6.6 REAL INCOME

As discussed earlier in this chapter, index numbers are very useful in finding the real income of an individual or a group of them, which facilitates the different managements to decide their wage policies. The

process of measuring the actual income vis-a-vis the changes in prices is called as *deflation*.

The formula for computing the real income is as follows:

$$\text{Real Income of a year} = \frac{\text{Money Income for the year}}{\text{Price Index of that year}} \times 100$$

Example 12

Calculate the real income for the following data:

Year	1990	1991	1992	1993	1994	1995
Income in Rs.	800	1050	1200	1600	2500	2800
Price Index	100	105	115	125	130	140

Solution: The real income is calculated by the formula:

$$\text{real income} = \frac{\text{Money Income for the year}}{\text{Price Index of that year}} \times 100$$

The table of computation of real income's is completed as shown below:

Year	Income in Rs.	Price Index	Real Income
1990	800	100	800
1991	1050	105	$\frac{1050}{105} \times 100 = 1000$
1992	1200	115	$\frac{1200}{115} \times 100 = 1043$
1993	1600	125	$\frac{1600}{125} \times 100 = 1280$
1994	2500	130	$\frac{2500}{130} \times 100 = 1923$
1995	2800	140	$\frac{2800}{140} \times 100 = 2000$

6.7 DEMERITS OF INDEX NUMBERS

- (1) There are numerous types and methods of constructing index numbers. If an appropriate method is not applied it may lead to wrong conclusions.
- (2) The sample selection may not be representative of the complete series of items.
- (3) The base period selection also is personalized and hence may be biased.
- (4) Index number is a quantitative measure and does not take into account the qualitative aspect of the items.
- (5) Index numbers are approximations of the changes, they may not accurate.

Check Your Progress

1. Define Index Numbers.
2. Write a short note on the importance of Index Numbers.
3. "Index Numbers are the Economical barometers". Discuss this statement with examples.
4. Discuss the steps to construct Index Numbers.
5. What are the problems in constructing an Index Number?
6. Define Cost of Living Index Number and explain its importance.
7. What do you mean by (i) Chain Based Index Number and (ii) Fixed Base Index Number? Distinguish between the two.
8. Define (i) Laspeyre's Index Number, (ii) Paasche's Index Number and (iii) Fisher's Index Number. What is the difference between the three? Which amongst them is called as the ideal Index Number? Why?
9. What are the demerits of Index Numbers?
10. From the following data, construct the price index number by simple aggregative method:

Commodity	Unit	Price in	
		1990	1991
A	Kg	14	18
B	Kg	6	9
C	Litre	5	8
D	Litre	12	20

Ans: 148.65

11. From the following data, construct the price index number for 1995, by simple aggregative method, with 1994 as the base:

Commodity	Unit	Price in	
		1994	1995
Rice	Kg	8	10
Wheat	Kg	5	6.5
Oil	Litre	10	13
Eggs	Dozen	4	6

Ans: 131.48

12. From the following data, construct the price index number for 1986, by average of price relatives method:

Commodity	Unit	Price in	
		1985	1986
Banana	Dozen	4	5
Rice	Kg	5	6
Milk	Litre	3	4.5
Slice Bread	One Packet	3	4

Ans: 132.08

13. From the following data, construct the price index number, by method of average of price relatives:

Commodity	Unit	Price in	
		1988	1990
A	Kg	6	7.5
B	Kg	4	7
C	Kg	10	14
D	Litre	8	12
E	Litre	12	18

Ans: 148

14. From the following data, construct the price index number for 1998, by (i) simple aggregative method and (ii) simple average of price relatives method, with 1995 as the base:

Commodity	Unit	Price in	
		1995	1998
Rice	Kg	12	14
Wheat	Kg	8	10
Jowar	Kg	7	9
Pulses	Kg	10	13

Ans: (i) 124.32, (ii) 125.06

15. From the following data, construct the weighted price index number:

Commodity	A	B	C	D
Price in 1985	10	18	36	8
Price in 1986	12	24	40	10
Weight	40	25	15	20

Ans: 121.29

16. From the following data, construct the index number using (i) simple average of price relatives and (ii) weighted average of price relatives:

Commodity	Weight	Price in	
		1988	1990
Rice	4	8	10
Wheat	2	6	8
Pulses	3	8	11
Oil	5	12	15

Ans: (i) 130.13, (ii) 128.82

17. From the data given below, construct the Laspeyre's index number:

Commodity	1975		1976
	Price	Quantity	Price
A	5	10	8
B	6	15	7.5
C	2	20	3
D	10	14	12

Ans: 131.40

18. From the data given below, construct the Paasche's index number:

Commodity	1980		1985
	Price	Price	Quantity
A	4	7	10
B	14	22	16
C	5	7	30
D	8	10	21

Ans: 144.67

19. From the following data given below, construct the (i) Laspeyre's index number, (ii) Paasche's index number and hence (iii) Fisher's index number.

Commodity	1980		1990	
	Price	Quantity	Price	Quantity
A	6	15	9	21
B	4	18	7.5	25
C	2	32	8	45
D	7	20	11	29

Ans: (i) 203.83, (ii) 203.37, (iii) 203.60

20. From the following data given below, construct the (i) Laspeyre's index number, (ii) Paasche's index number and (iii) Fisher's index number.

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
Cement	140	200	167	254
Steel	60	150	95	200
Coal	74	118	86	110
Limestone	35	50	46	60

Ans: (i) 103.98, (ii) 127.4, (iii) 115.09

21. From the following data given below, construct the Fisher's index number:

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
A	2	8	4	14
B	6	14	7	20
C	8.5	10	12	15
D	14	8	19	12
E	22	60	38	85

Ans: 131.37, 120.15

22. From the following data, construct the aggregative price index numbers by taking the average price of the three years as base.

Commodity	Price in 1980	Price in 1981	Price in 1982
A	10	12	16
B	16	19	25
C	5	7	10

Ans: 81.58, 100, 134.21

23. From the following data, construct the price index number by taking the price in 1978 as the base price using aggregative method:

Commodity	Price in 1978	Price in 1979	Price in 1980
A	16	18	24
B	4	6	7.5
C	11	15	19
D	20	28	30

Ans: 131.37, 120.15

24. From the following data, construct the price index number by taking the price in 1998 as the base price:

Commodity	Weight	Price in 1998	Price in 1999	Price in 2000
A	3	12	15	20
B	1	8	9	11
C	4	16	20	25
D	2	15	18	22

25. From the following data, construct (i) I_L , (ii) I_P , (iii) I_F

Commodity	1969		1970	
	Price	Quantity	Price	Quantity
Rice	2	10	3	12
Wheat	1.5	8	1.9	10
Jowar	1	6	1.2	10
Bajra	1.2	5	1.6	8
Pulses	4	14	6	20

Ans: 144.4, 144.56, 144.28

26. Construct the cost of living index number for 1980 using the Family Budget Method:

Item	Quantity	Price in	
		1975	1980
A	10	5	7
B	5	8	11
C	7	12	14.5
D	4	6	10
E	1	250	600

Ans: 192.95

27. Construct the cost of living index number for the following data with base year as 1989.

Item	Weight	Price in		
		1989	1990	1991
Food	4	45	50	60
Clothing	2	30	33	38
Fuel	1	10	12	13
House Rent	3	40	42	45
Miscellaneous	1	5	8	10

Ans: for 1990: 114.49, **for 1991:** 132.20

28. A survey of families in a city revealed the following information:

Item	Food	Clothing	Fuel	House Rent	Misc.
% Expenditure	30	20	15	20	15
Price in 1987	320	140	100	250	300
Price in 1988	400	150	125	250	320

What is the cost of living index number for 1988 as compared to that of 1987?

Ans: 113.65

29. Construct the consumer price index number for the following industrial data:

Item	Weight	Price Index
Industrial Production	30	180
Exports	15	145
Imports	10	150
Transportation	5	170
Other activity	5	190

Ans: 167.30

30. Calculate the real income for the following data:

Year	1988	1989	1990	1991	1992	1993
Income in Rs.	500	550	700	780	900	1150
Price Index	100	110	115	130	140	155

31. The employees of Australian Steel ltd. have presented the following data in support of their contention that they are entitled to a wage adjustment. Dollar amounts shown represent the average weekly take home pay of the group:

Year	1973	1974	1975	1976
Pay in \$	260.50	263.80	274	282.50
Index	126.8	129.5	136.2	141.1

Compute the real wages based on the take home pay and the price indices given. Also compute the amount of pay needed in 1976 to provide buying power equal to that enjoyed in 1973.

32. Calculate the real income for the following data:

Year	1977	1978	1979	1980	1981	1982
Income in Rs.	250	300	350	500	750	1000
Price Index	100	105	110	120	125	140

33. The per capita income and the corresponding cost of living index numbers are given below. Find the per capita real income:

Year	1962	1963	1964	1965	1966	1967
per capita income	220	240	280	315	335	390
cost of living I.N.	100	110	115	135	150	160

34. The following data gives the salaries (in '00 Rs.) of the employees of Hindusthan Constructions Ltd with the cost of living index number. Find the real income and suggest how much allowance should be paid to them to maintain the same standard of living.

Year	1990	1991	1992	1993	1994	1995
Income	12	14	17	20	24	28
Price Index	100	120	135	155	180	225

35. The income of Mr. Bhushan Damle in 1999 was Rs. 8,000 per month. If he gets an increment of Rs. 1,200 in 2000 and the price index being 115 with base as 1999, can you conclude that Mr. Damle has got an increment which will maintain his standard of living as compared with the previous year?



Unit - 7

PROBABILITY DISTRIBUTIONS

Unit Structure :

- 7.1 Introduction
- 7.2 Mathematical Expectation and Variance
- 7.3 Binomial Distribution
- 7.4 Normal Distribution

7.1 INTRODUCTION

In the previous chapter we have seen that outcomes of an experiment can be expressed in numbers. In an experiment of throwing a die, the possible outcomes are expressed as 1, 2, 3, 4, 5, or 6. In experiments like tossing a coin or picking a card or drawing a ball from a bag, the outcomes are numbers. But they can be assigned values like in tossing a coin, the outcome of heads can be assigned value 0 and that of a tails can be assigned value 1. Thus, in some way all points in a sample space can be assigned numerical values.

A relation which assigns every outcome of an experiment to a real number is called as a random variable also called as stochastic variable. We can also say thus, that a random variable assures the probability for every outcome of an experiment.

Example 1:

In the above example of throwing a die, the random variable say X takes values $\{1, 2, 3, 4, 5, 6\}$.

In an experiment of tossing two coins, we can assume the random variable as the number of heads (or tails) in an outcome. The sample space is $S = \{HH, HT, TH, TT\}$. If number of heads denotes the value of the random variable (X), then the first outcome has 2 heads, second and third has 1 heads and fourth outcome has no heads. Thus, $X = \{2, 1, 1, 0\}$. We will have a different random variable (Y) if we take the number of tails as the counter for the random variable. In that case $Y = \{0, 1, 1, 2\}$.

Discrete random variable:

A random variable which takes discrete (distinct) values or to say in mathematical words as the variable which takes finite or countably infinite values is called as *discrete random variable*.

Example 2:

The scores on a die, the number of calls received at a call centre, number of letters typed by a secretary, number of strikes in a factory etc are examples of discrete random variables.

Probability Distribution:

We know that with every value of the random variable (X) there is a probability ($P(X)$) assigned for that particular outcome. The set of all values of the random variable along with their corresponding probabilities is called as *the probability distribution* of the random variable.

Example 3:

In an experiment of tossing three coins, if the random variable X denotes the number of heads in every outcome then the probability distribution table of X is as shown below:

X	0	1	2	3
$P(X)$	1/8	3/8	3/8	1/8

In an experiment of throwing two dice, if the random variable represents the sum of the scores on the upper faces of both the dice then its probability distribution table is as shown below:

X	Events	$P(X)$
2	(1, 1)	1/36
3	(1, 2), (2, 1)	2/36 = 1/18
4	(1, 3), (2, 2), (3, 1)	3/36 = 1/12
5	(1, 4), (2, 3), (3, 2), (4, 1)	4/36 = 1/9
6	(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)	5/36
7	(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)	6/36 = 1/6
8	(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)	5/36
9	(3, 6), (4, 5), (5, 4), (6, 3)	4/36 = 1/9
10	(4, 6), (5, 5), (6, 4)	3/36 = 1/12
11	(5, 6), (6, 5)	2/36 = 1/18
12	(6,6)	1/36

The events are explicitly written to count the probability. It is not expected to write the events every time. One should also observe that the total of the column of probabilities is 1, which we already know from the previous chapter that $\sum P(A_i) = 1$ for all events A_i .

The function $P(X)$ is called the *probability function* of X .

The probability distribution of a random variable is called as *discrete probability distribution* and the corresponding probability function is called as *probability mass function*.

7.2 MATHEMATICAL EXPECTATION and VARIANCE

Let X be a random variable taking values x_1, x_2, \dots, x_n with corresponding probabilities p_1, p_2, \dots, p_n respectively. The *Expectation* of x , denoted as $E(x)$ is given by the formula:

$$E(x) = p_1x_1 + p_2x_2 + \dots + p_nx_n = \sum p_i x_i, \text{ where } \sum p_i = 1$$

In cases of games of chance if the amount received by a player if he wins a game is a , the amount he loses if he does not win the game is b and the probability of winning the game is p then the expectation is given by the formula: $ap - bp'$, where $p' = 1 - p$. The negative sign because of the loss the person suffers.

The expectation of a random variable is called as its *mean*. The mean of a random variable is denoted by the Greek letter μ (pronounce as 'mu' of music).

Laws of Expectation

If X and Y are two random variables then

1. $E(X) \geq 0$.
2. The expected value of their sum $X + Y$ is given by : $E(X \pm Y) = E(X) \pm E(Y)$.
3. Similarly, $E(aX \pm bY) = aE(X) \pm bE(Y)$.
4. The expected value of their product XY is given by: $E(XY) = E(X).E(Y)$

Variance of a random variable

Variance of a random variable X denoted by $V(X)$ is the square of the standard deviation of X and is calculated by the formula: $V(X) = E(X^2) - [E(X)]^2$

Example 4:

A die is thrown at random. What is the expectation and variance of the number on it?

Ans: When a die is thrown the possible outcomes are $X = \{1, 2, 3, 4, 5, 6\}$ with each having a probability of $1/6$. This can be tabulated as follows:

x_i	1	2	3	4	5	6
p_i	1/6	1/6	1/6	1/6	1/6	1/6

$$\therefore E(X) = p_1x_1 + p_2x_2 + \dots + p_nx_n = (1 + 2 + 3 + 4 + 5 + 6) \times 1/6 = 21/6 = 3.5$$

$$\text{Now, } E(X^2) = 1 \times 1/6 + 2^2 \times 1/6 + 3^2 \times 1/6 + 4^2 \times 1/6 + 5^2 \times 1/6 + 6^2 \times 1/6 \\ = (1 + 4 + 9 + 25 + 36) \times 1/6 = 75/6 = 12.5$$

$$V(X) = E(X^2) - [E(X)]^2 = 12.5 - (3.5)^2 = 12.5 - 12.25 = 0.25$$

Example 5:

Three coins are tossed. What is the expectation of a heads occurring?

Ans: When three coins are tossed the sample space is as follows:

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

The outcomes for a head are: 0 heads, 1 heads, 2 heads and 3 heads.

The corresponding probabilities are shown below:

x_i (heads)	0	1	2	3
p_i	1/8	3/8	3/8	1/8

$$\therefore E(\text{no. of heads}) = E(x) = \sum p_i x_i = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} = \frac{12}{8}$$

$$\therefore E(x) = 1.5$$

Example 6:

The probability that it will rain on a day is 0.6. Find the expectation of umbrella seller who earns a profit of Rs. 3050 if it rains and a loss of Rs. 250 if it does not rain on a day.

Ans: Given $p = 0.6 \therefore p' = 1 - 0.6 = 0.4$, $a = 3050$ and $b = 250$

\therefore the mathematical expectation = $3050 \times 0.6 - 250 \times 0.4 = 1830 - 100 = 1730$

Exercise

- (1) Define (a) random variable, (b) discrete random variable, (c) probability mass function
- (2) Define (a) mean and (b) of a random variable.
- (3) Prepare a frequency distribution table of tossing 3 coins.
- (4) Prepare a frequency distribution table of throwing two dice.
- (5) If it rains, a dealer in umbrella can earn Rs. 300 per day and if it does not rain he can lose Rs.80 per day. What is the expectation if the probability of a rainy day is 0.57?
- (6) A person plays a game where he earns Rs. X^2 if he gets X on a die. Find the mean and variance of his earning.
- (7) In a game, a person X throws a coin three times. He is paid Rs. 400 if he gets a heads all three times. The entry fee for the game is Rs. 80. What is the mathematical expectation of X ?
- (8) A shop owner earns a profit of Rs. 1000 per day. On a holiday he suffers a loss of Rs. 300 per day. If the probability that a day is a holiday is 0.14, find his mathematical expectation.
- (9) A die is thrown at random. What is the expectation of the number on it?
- (10) If two dice are thrown, what is the expectation of the sum of the sample points?
- (11) If n dice are thrown, what is the expectation of the sum of the sample points?

(Hint: In above two problems use laws of expectation)

- (12) Find the mean and variance when a coin is tossed two times.
- (13) If three coins are tossed, what is the expectation and variance of the number of tails?
- (14) A person draws 2 balls from a bag containing 6 white and 5 black balls. He is paid Rs. 22, if he draws both balls of white color and Rs. 11 if he draws one of each color. Find his expectation.
- (15) A person draws 3 balls from a bag containing 3 white, 4 red and 5 black balls. He is offered Rs. 10, Rs. 5 and Rs. 2 if he draws three balls of same color, 2 balls of same color and 1 ball of each color respectively. Find his expectation.
- (16) The probability there is at least one error in accounts statement prepared by A is 0.25, by B is 0.35 and by C is 0.4. If A, B and C prepared 10, 16 and 20 statements respectively, find the expected number of error free statements.
- (17) Three Mathematics teachers X, Y and Z were given 120, 200 and 150 papers of an examination to assess respectively. The probability that there is totaling mistake in a paper by A, B and C is 0.2, 0.55 and 0.25 respectively. Find the expected number of (a) papers in which error is possible and (b) error free papers.
- (18) If X and Y are two independent discrete random variables such that $E(X) = 12$, $E(Y) = 20$, then find the expectation of $Z = 2X + 3Y$.
- (19) In a lottery game there are 10 tickets. 3 tickets have a prize of Rs. 2, 2 tickets have a prize of Rs. 5 and 1 ticket has a prize of Rs. 10. The remaining tickets are blank. Find the expectation of a player winning a prize.
- (20) A newspaper agent earns Rs.200 a day if there is some breaking news in it and loses Rs. 20 a day if there is no breaking news. The probability that there is a breaking news in the paper is 0.45. Find the expectation of his earnings.
- (21) A gambler draws a card from a pack of 52 cards. He earns Rs. 104 if it is an ace, Rs. 52 if it is a King or a Queen, Rs. 26 if it is a Jack and loses Rs. 13 if it is any other card. Find his expectation.

(22) Find the mean and variance for the following probability distribution:

X	- 5	0	5	10	15
$P(X)$	1/3	1/9	1/3	1/9	1/9

(23) Find the mean and variance for the following probability distribution:

X	0	2	4	6	8	10
$P(X)$	0.05	0.15	0.02	0.28	0.3	0.2

(24) The probability distribution of daily demand of cell phones in a mobile gallery is given below. Find the mean and variance.

Demand	5	10	15	20
Probability	0.4	0.22	0.28	0.10

(25) The probability distribution of number of divorce cases withdrawn per day from a court is given below. Find the mean and variance.

No. of Cases	0	1	2	3	4	5
Probability	0.2	0.26	0.12	0.22	0.08	0.12

(26) At the famous 'Jumbo Vada' shop near Dadar station in Mumbai, the probability distribution of arrival of number of customers per minute is given below. Find the expectation of customers per minute.

No of Customers	0	2	4	6	8
Probability	0.24	0.22	0.26	0.18	0.1

(27) Mr. Chinmay has bought a new motorcycle from 'Swastik Agency'. The agency offers after sales service contract for Rs. 600 for four years. From a market survey the probability distribution of the expenses on service in four years for the same brand is known and is given below. Should Mr. Chinmay pay for the service contract?

Expenses	200	400	600	800	1000	1200
Probability	0.34	0.22	0.1	0.08	0.22	0.04

7.3 BINOMIAL DISTRIBUTION

In many situations we see that a same experiment is repeated number of times. Also the probabilities of the outcomes are fixed irrespective of the previous trials. Let us consider the example of tossing a coin three times. Here one experiment of tossing a coin is done three times and the probability of getting a head or getting a tail is always the same *i.e.* $1/2$. Let us call the event of getting a heads as a 'success' and getting a tails as a 'failure' and the corresponding probabilities as p and q where $p + q = 1$ (or $q = 1 - p$). Consider an outcome 'HTT'. With our notations just introduced, there is one success and 2 failures in this outcome. The probability of this outcome is obtained by multiplying the individual probabilities. So we have the probability of HTT as $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ or $p \times q \times q = pq^2$. But if the order is not important then the probability of getting one success and two failures is $3 \times pq^2$, as there are three such outcomes *viz* HTT, THT and TTH. It is easy to do so if the number of trials is less.

When the number of trial is more and the probabilities of outcomes are fixed either a success or a failure, Bernoulli characterized such probability distributions and called them as *Binomial Distributions*.

Characteristics of Binomial Distributions

1. The experiment consists of n (a finite number) number of trials.
2. The outcome of every trial is either a success or a failure and is independent of the previous trial(s).
3. The probabilities of success (and hence failures) remains constant for every trial.
4. If p denotes the probability of success, $q = 1 - p$ denotes the probability of failure and r denote the number of successes in all n trials then the probability of r number of successes is given by the formula: $P(X = r) = {}^n C_r p^r q^{n-r}$. This is the probability mass function of r . Here $r = 0, 1, 2, 3, \dots, n$.
5. The probabilities given in the formula are nothing but the terms in the binomial expansion of $(p + q)^n$. Hence the name given to this distribution is Binomial Distribution.
6. Mean of a Binomial Distribution is given by $\mu = np$.
7. Variance of a Binomial Distribution is given by npq . Hence its standard deviation is $\sigma = \sqrt{npq}$.

Example 7:

The mean of a binomial distribution is 12 and standard deviation is

3. Calculate n , p and q .

Ans: Given $np = 12$, $\sigma = \sqrt{npq} = 3 \Rightarrow npq = 9$

$$\therefore 12 \times q = 9 \Rightarrow q = 9/12 = 3/4$$

$$\therefore p = 1 - q = 1 - 3/4 = 1/4$$

$$\therefore n \times 1/4 = 12 \Rightarrow n = 12 \times 4 = 48$$

$$\text{Thus, } n = 48, p = 1/4 \text{ and } q = 3/4$$

Example 8:

The mean and variance of a binomial distribution are 14 and 9.

Comment on this statement.

Ans: From the given information, $np = 14$ and $\sigma = \sqrt{npq} = 9 \Rightarrow npq = 81$

$$\therefore 14 \times q = 81 \Rightarrow q = 81/14 = 5.78$$

We know that probability of any outcome is never greater than 1. Thus, the information given is inconsistent.

Example 9

The probability that a youth exercises every day is 0.6. Find the probability that out of 5 youths selected (i) none of them do exercise, (ii) atleast one exercises.

Ans: Given $p = 0.6 \Rightarrow q = 1 - p = 1 - 0.6 = 0.4$ and $n = 5$

(i) To find probability that none of them exercise means $r = 0$

We know that: $P(X = r) = {}^n C_r p^r q^{n-r}$

$$\therefore P(X = 0) = {}^5 C_0 (0.6)^0 (0.4)^{5-0} = (0.4)^5 = 0.01024$$

(ii) To find probability of at least one does exercise. This is a complementary event of no one does exercise, whose probability we have already found.

$$P(\text{at least one does exercise}) = 1 - P(X = 0) = 1 - 0.01024 = 0.98976$$

Example 10

The average rainfall in a 30 days' month is 50%. Find the probability that (i) the first four days of a given week will be fine and the remaining will be wet, (ii) rain will fall on just three days of a week.

Ans: Given $p = 50\% = 0.5 \Rightarrow q = 0.5$

(i) To find probability that the first four days of a given week will be fine and the remaining will be wet. Here the days of rainfall is fixed.

$$\therefore \text{probability for rainfall in remaining days of a week} = (0.5)^3 (0.5)^4 = 0.0078$$

(ii) To find the probability that rain will fall on just three days of a week. Here which three days is specified.

$$\therefore P(X = 3) = {}^7 C_3 (0.5)^3 (0.5)^4 = \frac{7 \times 6 \times 5}{3 \times 2} \times 0.0078 = 0.27$$

Example 12

The food inspector along with his colleagues comes for an inspection of a drug. The number of faulty drug tablets found in every sample pack of 16 tablets is given below. The following table shows the distribution of 160 tablets.

No of faulty tablets	0	1	2	3	4	5
No of samples	26	22	30	35	36	11

(i) Fit a binomial distribution and find the expected frequencies, if the chance of the machine being defective is $\frac{1}{2}$

(ii) Find the mean and standard deviation of the fitted distribution.

Ans: (i) Given $p = \frac{1}{2} \Rightarrow q = \frac{1}{2}$, $n = 5$ and $N = 160$

We will now find the probabilities of the number of faulty tablets by finding the binomial expansion of $(p + q)^5$.

$$\text{Now, } (p + q)^5 = 1p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + 1q^5$$

Substituting values of p and q . we have

$$(p + q)^5 = \frac{1}{32} + \frac{5}{32} + \frac{10}{32} + \frac{10}{32} + \frac{5}{32} + \frac{1}{32}$$

Now, the numbers $\frac{1}{32}, \frac{5}{32}, \frac{10}{32}, \frac{10}{32}, \frac{5}{32}, \frac{1}{32}$ are the corresponding probabilities of 0, 1, 2, 3, 4 and 5 no. of faulty tablets.

The expected frequencies are calculated by multiplying each probability with $N = 160$

Thus, the observed frequencies with the expected frequencies are tabulated as shown below:

No of faulty tablets	0	1	2	3	4	5
Observed Frequency	26	22	30	35	36	11
Expected Frequency	5	25	50	50	25	5

$$(ii) \text{ Mean} = np = 5 \times \frac{1}{2} = 2.5$$

$$\text{standard deviation} = \sigma = \sqrt{npq} = \sqrt{5 \times \frac{1}{2} \times \frac{1}{2}} = 1.12$$

Exercise

- (1) If $n = 10$ and $p = \frac{1}{4}$, find the mean and variance of the binomial distribution.
- (2) The mean of a binomial distribution is 40 and standard deviation is 6. Calculate n , p and q .
- (3) The mean of a binomial distribution is 6 and its standard deviation is $\sqrt{2}$. Find n , p and q .
- (4) The mean of a binomial distribution is 3 and the variance is 1.2, find n and p .
- (5) The mean of a binomial distribution is 5 and variance is $\frac{5}{2}$. Find $P(X = 4)$.
- (6) The mean and variance of a binomial distribution are 3 and $\frac{6}{5}$. Find $P(X > 3)$.
- (7) If $n = 8$ and $p = \frac{2}{3}$, find the mean and standard deviation of the binomial distribution.
- (8) Comment on “The mean and variance of a binomial distribution are 4 and 6”.
- (9) Comment on “The mean and standard deviation of a binomial distribution are 6 and 4”.
- (10) A bag contains 10 white and 5 black balls. If 4 balls are selected at random, find the probability that (i) 3 are black balls, (ii) atleast one black ball is selected.
- (11) Out of the total passengers travelling by BEST buses 40% do not have the exact money for their tickets. If a conductor gives tickets to

- 15 passengers, find the probability that less than 5 passengers will pay exact money.
- (12) In a government department an officer is not in his chair 12 out of 30 days in a month. If the senior officer investigates his office 6 times, what is the probability that he is not in his chair 3 times.
- (13) The probability that internet users will buy a product from an online marketing advertisement is 0.35. If 100 users login the site, find the probability that 15 or more than that would actually buy the product?
- (14) The EGS of central government guarantees employment to every 4 out of 6 unemployed persons. If 100 people are selected at random from a village, find the probability that 10 of them have received employment.
- (15) The probability that a student from distance education passes is 0.6. Find the probability that out of 6 students selected (i) no student passes, (ii) atleast one student passes.
- (16) The probability that an evening college student will graduate is 0.4. Determine the probability that out of 5 students (i) none, (ii) one, (iii) atleast one student will graduate.
- (17) The overall passing percentage of students enrolled in a distance education institute is 35%. Find the probability that out 6 students 3 students pass.
- (18) The probability that a micro wave oven is found to be defective is 0.2. If 6 microwave ovens are selected, find the probability that all are defective.
- (19) The probability of a defective bolt is $\frac{1}{10}$. Find the mean and variance of defective bolts out of a total of 400 bolts.
- (20) 5 out of 25 items are found to be defective. If 4 items are selected find the probability distribution of the defective items.
- (21) 6 out of 50 items in a lot are found to be defective. Find the probability of the following if 4 items are selected: (i) one defective item, (ii) 3 defective items, (iii) at most 3 defective items.
- (22) 6% of articles in a given bundle are defective. Find the probability that in a sample of 5 articles, none is defective.
- (23) The probability that a long range missile hits a target is 0.8. If 4 missiles are shot, find the probability that (i) exactly two will hit the target, (ii) atleast two will hit the target.
- (24) The probability of a man hitting a target is $\frac{1}{4}$. If h fires 7 times, what is the probability that he hits the target atleast twice?
- (25) The average rainfall in a 30 days' month is 40%. Find the probability that (i) the first four days of a given week will be fine and the remaining will be wet, (ii) rain will fall on just three days of a week.
- (26) Out of 1000 families with 4 children each, how many would you expect to have (i) 2 boys and 2 girls, (ii) atleast one boy, (iii) no girl

- (27) If on an average rain falls on 10 days in every 30 days, find the probability that (i) the first three days are fine and the remaining are wet, (ii) the rain will fall on just three days of a week
- (28) A box contains 60 Alphanso mangoes of which 8 are normal mangoes. 8 mangoes are selected by the food inspector. Find the probability that (i) 6 mangoes are normal type, (ii) atleast one is an Alphanso type mango.
- (29) The incidence of a certain disease is such that on an average 20% of workers suffer from it. If 10 workers are selected at random, find the probability that (i) exactly 2 workers suffer from the disease, (ii) not more than 2 workers suffer from the disease, (iii) not more than 2 workers suffer from the disease.
- (30) An unbiased die is tossed 3 times. Find the probability of obtaining (i) no six, (ii) all sixes.
- (31) An unbiased die is tossed 4 times. Find the probability of obtaining (i) atleast one six, (ii) 4 sixes.
- (32) Four coins are tossed simultaneously. What is the probability of getting (i) 4 heads, (ii) 2 heads, 2 tails, (iii) atleast one heads
- (33) Five coins are tossed 3200 times. Find the frequency distribution of heads and tails. Also find the mean and variance.
- (34) 8 coins are tossed 256 times. Find the expected frequencies of success. Also find the mean and variance of the fitted values.
- (35) The screws produced by a certain machine were checked by examining the number of defectives in a sample of 12. The following table shows the distribution of 128 samples.

(36)

No of defectives	0	1	2	3	4	5	6	7
No of samples	7	6	19	35	30	23	7	1

- (i) Fit a binomial distribution and find the expected frequencies, if the chance of the machine being defective is 0.5
- (ii) Find the mean and standard deviation of the fitted distribution.

7.4 NORMAL DISTRIBUTION

The Binomial distribution is a discrete probability distribution as the random variable considered is a discrete. But for many practical problems related to sales volume, height, weight of an individual or a product, length of an item, strength, resistance, life of an electrical instrument the random variable is of continuous type

Normal distribution is the most commonly used continuous probability distribution. The probability density function of a normal distribution is:

$$P(X) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{X-\mu}{\sigma}\right]^2} \quad \text{for } -\infty < X < \infty$$

Here μ is the mean and σ is the standard deviation of the distribution.

Characteristics of a Normal curve

Fig 7.1

1. The curve of the normal curve is bell shaped.
2. The curve is symmetrical about the central vertical line corresponding to the mean μ of the distribution.
3. The peak of the curve is obtained at $X = \mu$.
4. All the three measures of central tendencies mean, median and mode coincide for a normal curve.
5. There are two tails of the curve which extend infinitely in both positive and negative X-axis and never touch the axis.
6. The area under the normal curve is unity *i.e.* 1. This is because the area represents the probabilities of the variable and the sum of all probabilities as we know is 1. Due to symmetry, the area to the left of the mean is exactly 50% *i.e.* 0.5 and similarly the area to the right of the mean is also 0.5. Thus, for a normal curve, $P(X \leq \mu) = P(X \geq \mu)$.
7. The standard deviation determines the spread of the distribution around the mean. As shown in Fig 9.2, if the value of σ is small then the curve will be narrow and if the value of σ is large the curve will become wider indicating the deviations of data around the mean.
8. The area covered between $\mu \pm \sigma$ is 68.26% of the total area under the normal curve.
9. The area covered between $\mu \pm 2\sigma$ is 95.44% of the total area under the normal curve.
10. The area covered between $\mu \pm 3\sigma$ is 99.74% of the total area under the normal curve.

Fig 7.2

Area under the Normal Curve

If X is a random normal variable, then the formula to calculate the probability function uses methods of integration which are quite cumbersome. This lead to a procedure of defining a new variable $z = \frac{X - \mu}{\sigma}$, called as the standard normal variate (S.N.V.). It is observed that the mean of this SNV is zero and its standard deviation is unity. Now the problem of finding the probabilities for $X = X_1$, is equated with finding the area under the standard normal curve at $z = z_1$. The area between $z = 0$ and

$z = z_1$ is read from the standard normal tables. This table is provided in the appendix.

Now let us see some examples regarding how to read the standard normal tables:

Example 13:

To find the area under the curve when $z = 1.35$

Ans: We look at the first column of the table for the number 1.3 and then move horizontally till the column of 0.05 which corresponds to our 1.35 ($1.3 + 0.05$). The value in this cell is 0.4115. Thus, the area under the curve for $z = 1.35$ is 0.4115. This also means that $z = 1.35$ represents 41.15% of the total area.

Fig 7.3

Example 14:

To find area under the curve for $z > 1.62$

Ans: Repeating the previous step we find the value of $1.6 + 0.02$ from the table, this is 0.4474.

Now, we want to find area for $z > 1.62$. What we have got is the area between $z = 0$ and $z = 1.62$. The total area to the right of $z = 0$ is 0.5. Thus the required area is $0.5 - 0.4474 = 0.0526$

Fig 7.4

Example 15:

To find area under the normal curve between $z > -1.25$

Ans: We know that the normal curve is symmetric about the mean. So we repeat the same steps as described above to find the area between $z = 0$ and $z = 1.25$. From the table this area is 0.3944. Now, the area representing $z > -1.25$ is the area between 0 to -1.25 plus the remaining 50% area *i.e.* 0.5

Thus, the required area is $0.3944 + 0.5 = 0.8944$

Fig 7.5

Example 16

To find area between $-1.2 \leq z \leq 2$

Ans: The required area is split into two areas:
 $-1.2 \leq z \leq 0$ and $0 \leq z \leq 2$

The first area as we have already seen how to calculate is 0.3849 and the second area is 0.4772

Thus, the required area is a sum of these two areas which is 0.8621

Now let us solve some actual problems using our knowledge of reading the standard normal table values.

Fig 7.6

Example 17:

The heights of 1000 students in a college are normally distributed with mean 160 cm and standard deviation 12 cm. How many students will be there such that (i), their heights are greater than 165 cm (ii) their heights are between 150 cm and 170 cm, (iii) heights are less than 145cm.

Ans: Given: $N = 1000$, $\mu = 160$, $\sigma = 12$

(i) $X = 165$

$$z = \frac{X - \mu}{\sigma} = \frac{165 - 160}{12} = 0.42$$

$$P(X > 165) = P(z > 0.42)$$

The area under the normal curve represented by $z = 0$ and $z = 0.42$ is 0.1628

The probability of students with heights greater than 165 cm is 16.28% and the number of students are $1000 \times 0.1628 = 162.8 \approx 163$

(ii) $X_1 = 150$ and $X_2 = 170$

$$\therefore z_1 = \frac{X_1 - \mu}{\sigma} = \frac{150 - 160}{12} = -0.83$$

$$\text{and } z_2 = \frac{X_2 - \mu}{\sigma} = \frac{170 - 160}{12} = 0.83$$

The required probability is $P(X_1 \leq X \leq X_2)$

$$\begin{aligned} &= P(-0.83 \leq z \leq 0.83) \\ &= P(-0.83 \leq z \leq 0) + P(0 \leq z \leq 0.83) \\ &= P(0 \leq z \leq 0.83) + P(0 \leq z \leq 0.83) \\ &= 2 P(0 \leq z \leq 0.83) \end{aligned}$$

From the table we have the required probability as $2 \times 0.2967 = 0.5934$

Thus, the number of students whose heights are between 150cm and 170 cm is

$$1000 \times 0.5934 = 593.4 \approx 593.$$

(iii) $X = 145$

$$z = \frac{X - \mu}{\sigma} = \frac{145 - 160}{12} = -1.25$$

The area represented by $z \leq -1.25$,

Now, the area between $z = 0$ and $z = 1.25$ is 0.3944

\therefore the required area is $0.5 - 0.3944 = 0.1056$

\therefore the number of students with heights less than 145cm are 106.

Fig 7.7

Example 18:

If 2.28% of teachers in a State have salary less than Rs. 4,500 and 30.85% teachers have salary greater than Rs. 7,000. Find the average salary and standard deviation.

Ans: (In such kind of problems we do not find the area but we find the value of z corresponding to the area known. In simple words, we look inside the table and find the corresponding z value from the first column and first row.)

Since 2.28% of teachers have salary less than 4500, the remaining 47.72% lie to the right of $X = 4500$. The area to the right of $X = 4500$ is $0.5 - 0.028 = 0.4772$. Now look in the table, to find z -value for this is -2 . (As it is to the left of the mean)

$$z = \frac{4500 - \mu}{\sigma} = -2 \quad \Rightarrow 4500 - \mu = -2\sigma \quad \dots (1)$$

Since 30.85% teachers have salary more than Rs. 7000, the area to the right of $X = 7000$ is 0.3085. So, the remaining area to the left of $X = 7000$ is $0.5 - 0.3085 = 0.1915$

The z -value corresponding to this area is 0.5 from the table

$$\therefore z = \frac{7000 - \mu}{\sigma} = 0.5 \quad \Rightarrow 7000 - \mu = 0.5\sigma \quad \dots (2)$$

Solving (1) and (2) we have $\sigma = 1000$ and hence $\mu = 6500$. Thus, the mean salary is Rs. 6,500 and standard deviation is Rs. 1000

Exercise

- (1) Define (i) standard normal curve, (ii) Standard normal variate.
- (2) State the properties of standard normal curve.
- (3) Using the table of areas under a normal curve, find the probabilities of: (a) $P(0 \leq z \leq 1.2)$, (b) $P(0 \leq z \leq 2.3)$, (c) $P(-1 \leq z \leq 0)$, (d) $P(-1 \leq z \leq 1)$, (e) $P(-1.5 \leq z \leq 0.2)$, (f) $P(z \geq -1.6)$, (g) $P(z \geq 2)$.
- (4) If a random variate X is normally distributed with mean 45 and standard deviation 12, find the probabilities of the following: (a) $X \leq 30$, (b) $X \leq 55$, (c) $X \geq 60$, (d) $35 \leq X \leq 50$
- (5) If X is a normal random variable with mean 14 and standard deviation 6, find the probabilities of: (a) $X \leq 10$, (b) $X \leq 16$, (c) $X \geq 18$, (d) $10 \leq X \leq 20$.
- (6) If X_1 and X_2 are two random variates with means 30, 25 and standard deviations 16, 12 respectively, find $P(60 \leq Y \leq 80)$, where $Y = 3X_1 + 2X_2$.
- (7) The heights of 600 students in a college are normally distributed with mean 154 cm and standard deviation 16 cm. How many students will be there such that (i) their heights are between 140 cm and 160 cm,

- (ii) their heights are greater than 175 cm, (iii) heights are less than 135 cm.
- (8) The average life of a cartridge is 7 days with standard deviation 1.5 days. If the life expectancy of a cartridge shows a normal distribution, what is the probability that a cartridge functions for more than 10 days?
- (9) The weights of 1000 chocolates of a brand are normally distributed with mean weight as 15 mg and standard deviation 4mg. Find the number of chocolates with weights greater than 20mg.
- (10) The marks of 1200 students in a College show a normal distribution with mean 56 and standard deviation 6. Estimate the number of students with marks (i) less than 50, (ii) greater than 60, (iii) between 45 and 65.
- (11) The mean height of soldiers is 68.22 inches with variance 10.8. If the heights show a normal distribution, find the number of soldiers out of a regiment of 1000 whose height is greater than 6 feet.
- (12) The marks obtained by students are normally distributed with mean 65 and variance 25. What is the probability of students getting marks more than 75?
- (13) To pass a physical test for Air force, the height of a cadet should be at most 162cm. Out of 1400 cadets appeared for the physical test how many could not clear it, if their average height was 160 cm with standard deviation 9cm. Assume that the heights of the cadets show a normal distribution.
- (14) A manufacturer knows from his experience that the resistance of resistors he produces is normally distributed with mean 100 ohms and standard deviation 2 ohms. What is the percentage of resistors having resistance between 98 and 102 ohms?
- (15) The average life of a battery is 150 minutes with standard deviation 14 min. If the average life of battery shows a normal distribution, find the probability that a battery works for more than 180 min.
- (16) The mean and standard deviation of average monthly salaries of 6000 people are Rs. 20,000 and Rs. 3,500 respectively. Assuming that the data shows a normal distribution, find the (i) number of people with salaries greater than Rs. 25,000, (ii) salaries between Rs. 15,000 and Rs. 20, 000 and (iii) less than Rs. 10,000.
- (17) If 1.88% of teachers in a State have salary less than Rs. 3,500 and 4.85% teachers have salary greater than Rs. 8,000. Find the average salary and standard deviation.
- (18) In an examination marks obtained by students in Mathematics, Statistics and Economics are normally distributed with average marks 51, 53 and 46 and standard deviation 15, 12 and 16 respectively. Find the probability that the total marks are (i) 180 and above, (ii) less than 90.

- (19) The aggregate of students in a FYBSc class of a College is normally distributed with mean 425 marks and standard deviation 36 marks. A student is said to pass if he scores 40% of the total of 700 marks. Find the number of students (i) who have passed, (ii) whose aggregate is less than 325 marks.
- (20) The average diameter of a needle is measure as 0.1 mm with a standard deviation of 0.005mm. If 1600 samples are selected, find the number of needles with diameter (i) less than 0.08mm, (ii) greater than 0.2mm and (iii) between 0.6mm and 0.3mm
- (21) The daily sales of a firm are normally distributed with mean Rs. 8000 and standard deviation Rs. 100. What is the probability that on a certain day sales will be less than Rs. 8200 and what is the percentage of days with sales between Rs. 8050 and Rs. 8250?
- (22) 1000 light bulbs are installed in new factory, show a normal distribution with mean life of 120 days and standard deviation of 20 days. How many bulbs will expire in less than 90 days?
- (23) Record kept by goods inwards department of a large factory show that average number of lorries arriving each week is 248. It is known that the distribution is normal with standard deviation 26. If this pattern of arrival continues, what is the percentage of weeks expected to have the number of arrivals (i) less than 230 per week, (ii) more than 280 per week.
- (24) The income distribution workers in a certain factory were found to be normal with mean Rs. 500 and standard deviation Rs. 50. There were 228 persons with income above Rs. 600. How many workers were there in all?
- (25) The marks obtained by students in a subject showed a normal distribution with mean 65 and standard deviation 14. If there were 142 students with marks less than 50, find the total number of students.
- (26) For a normal distribution 30% items are below 45, 8% are above 64. Find the mean and variance.
- (27) In a normal distribution 10% item are under 35 and 89% under 63. Find the mean and standard deviation.
- (28) In a class, 30% students have marks less than 40, 33% have marks between 40 and 50 and the remaining have marks above 50. If the data is normally distributed, find the average marks and standard deviation.
- (29) A STD booth owner has an average balance of Rs. 230 and standard deviation of Rs. 30. Assuming that the balance with the owner behave normally, find the proportion of the balance being (i) less than Rs. 180 and (ii) between Rs. 200 and Rs. 250.
- (30) In an examination in a college 46% students secured a pass class and 10% students secured first class. If the minimum marks for pass class

and first class are 35 and 60 respectively. Find the average marks obtained by students.

- (31) The life of army shoes is normally distributed with mean 8 months and standard deviation 2 months. If 5000 pairs of shoes are issued, how many pairs would be expected to need replacement after 12 months?
- (32) The marks obtained by students in an examination are normally distributed with mean of 70 and standard deviation 6. If the top 5% students get grade A and the bottom 25% get grade F, what marks is the lowest A and the highest F?



QUESTION PAPER PATTERN**MARKS:- 100****TIME:- 3 HRS****N.B :**

- 1) All questions are compulsory
- 2) All question carry equal marks
- 3) Figures to the right indicate marks to a sub-question.
- 4) Graphs paper will be supplied on request.
- 5) Use of non-programmable calculator is allowed.

SECTION-I

- Q.1 Attempt any four of the following
(a) 5 marks (b) 5 marks (c) 5 marks
(d) 5 marks (e) 5 marks 20 marks
- Q.2 Attempt any four of the following
(a) 5 marks (b) 5 marks (c) 5 marks
(d) 5 marks (e) 5 marks 20 marks

SECTION-II

- Q.3 Attempt any four of the following
(a) 5 marks (b) 5 marks (c) 5 marks
(d) 5 marks (e) 5 marks 20 marks
- Q.4 Attempt any four of the following
(a) 5 marks (b) 5 marks (c) 5 marks
(d) 5 marks (e) 5 marks 20 marks
- Q.5 Attempt any four of the following
(a) 5 marks (b) 5 marks (c) 5 marks
(d) 5 marks (e) 5 marks 20 marks

